RYERSON UNIVERSITY

Department of Electrical and Computer Engineering

ELE 635
Communication Systems

Frequency Modulation

Winter 2015

- **Time- and frequency-domain description of angle modulated signals**
  - ❖ Phase Modulated (PM) signals
  - ❖ Frequency Modulated (FM) signals
  - ❖ Bandwidth of FM signals
- **Effects of nonlinearities on modulated signals**
  - ❖ Amplitude Modulated (AM) signals
  - ❖ Frequency Modulated (FM) signals
- **Generation of FM signals**
  - ❖ Indirect Method
- **FM Stereo Broadcasting**
  - ❖ Stereo signal multiplexing
  - ❖ Stereo signal demodulation
  - ❖ Tips, tricks, standards …

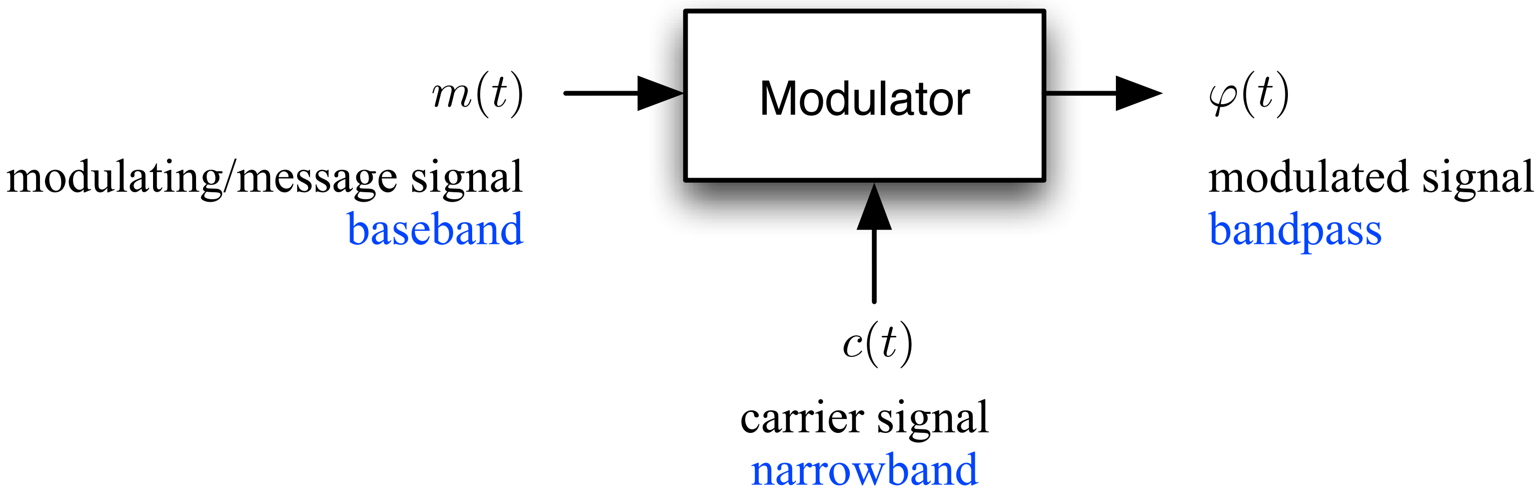
Time- and frequency-domain description

of

angle modulated signals

Modulation

$m(t)$ = modulating signal
$c(t)$ = carrier signal
$\varphi(t)$ = modulated signal



Modulation

$$\varphi(t) = a(t) \cos \theta(t)$$

(possibly) time-varying amplitude

(possibly) time-varying phase

$\varphi(t)$ represents a **rotating phasor** of:

- time-varying amplitude $a(t)$
- generalized phase $\theta(t)$
- instantaneous frequency $f_i(t)$:

$$f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \theta(t)$$

Unmodulated Carrier

An unmodulated carrier is of the form:

$$a(t) = A_c,$$
$$\theta(t) = 2\pi f_c t + \theta_0$$

such that:

$$\varphi(t) = A_c \cos\big(2\pi f_c t + \theta_0\big)$$

with:

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = f_c$$

- An unmodulated carrier has **constant amplitude** and **constant $f_i(t)$**.
- $\varphi(t)$ carries no information−all components are independent of $m(t)$.

Amplitude Modulation

Amplitude Modulation:

$$a(t) = g[m(t)]$$

$$\theta(t) = 2\pi f_c t + \theta_0$$

$$f_i(t) = f_c$$

such that:

$$\varphi_{AM}(t) = g[m(t)]\cos\left(2\pi f_c t + \theta_0\right)$$

Information contained in $m(t)$ is embedded in the **time-varying amplitude**.

Angle Modulation

Angle Modulation: either the phase or the frequency of the carrier wave is varied by $m(t)$ while the amplitude of the carrier wave is constant.

$$\varphi(t) = A_c \cos\theta(t)$$

where:

$$A_c = \text{constant}$$

$$\theta(t) = g\big[m(t)\big]$$

Information contained in $m(t)$ is embedded in the **time-varying phase**.

Angle Modulation

Phase Modulation (PM):

$$a(t) = A_c$$

$$\theta(t) = 2\pi f_c t + K_p m(t) + \theta_0$$

$$f_i(t) = f_c + \frac{K_p}{2\pi}\frac{d}{dt}m(t)$$

such that:

$$\varphi_{\mathrm{PM}}(t) = A_c \cos\big(2\pi f_c t + K_p m(t) + \theta_0\big)$$

Information contained in $m(t)$ is embedded in the generalized phase:

$\theta(t)$ is proportional to $m(t)$

Angle Modulation

Frequency Modulation (FM):

$$a(t) = A_c$$

$$\theta(t) = 2\pi f_c t + K_f \int_0^t m(\lambda) d\lambda + \theta_0$$

such that:

$$f_i(t) = f_c + \frac{K_f}{2\pi} m(t)$$

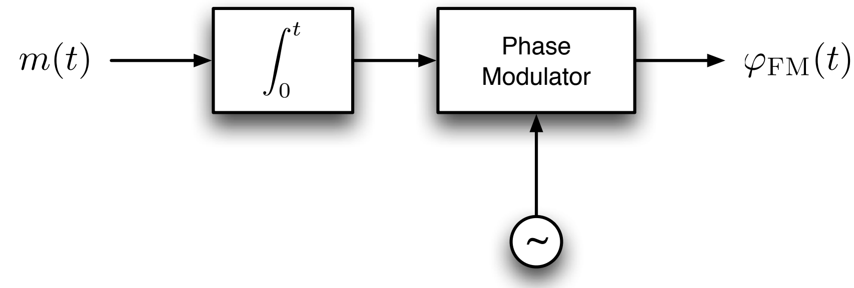
$$\varphi_{\mathrm{FM}}(t) = A_c \cos\big(2\pi f_c t + K_f \int_0^t m(\lambda) d\lambda + \theta_0\big)$$

Information contained in $m(t)$ is embedded in the instantaneous frequency:

$f_i(t)$ is proportional to $m(t)$

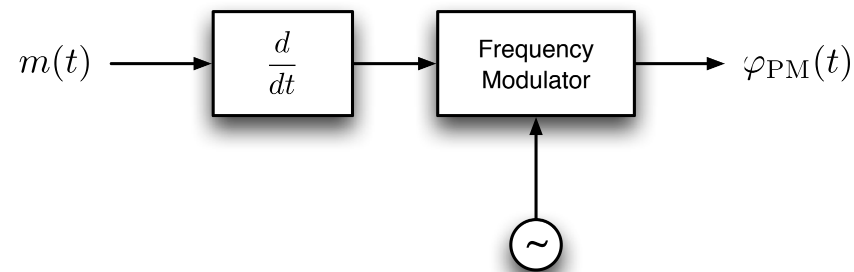
Angle Modulation

$$\theta(t) \sim \int_0^t m(\lambda)d\lambda$$



or we can consider PM as a special case of FM where:

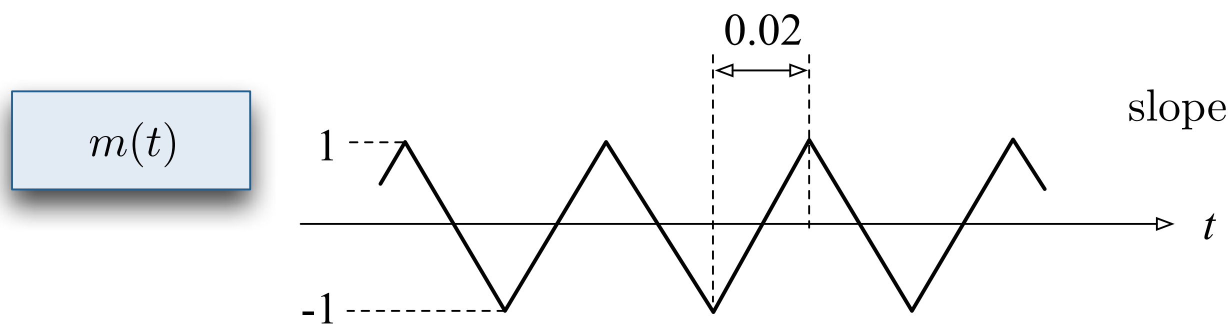
$$f_i(t) \sim \frac{d}{dt}m(t)$$



Need only one PM or FM → FM is the most common angle modulation; **Study FM**

Angle Modulation: An example

Consider the modulating waveform $m(t)$



$$\text{slope} = \pm\frac{2}{0.02} = \pm 100$$

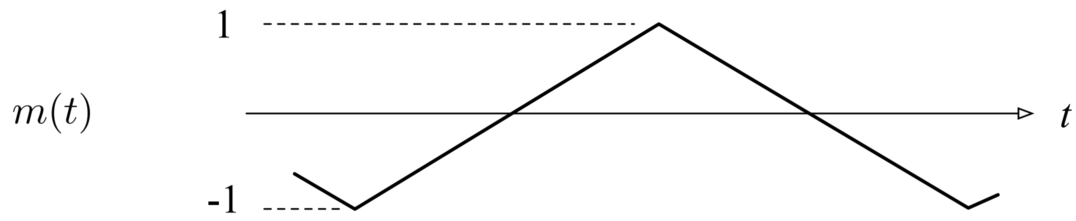
Determine the corresponding **PM** and **FM** waveforms for:

$K_f = K_p = 2\pi$ | $\theta_o = 0$ | $f_c = 1000$ Hz

Angle Modulation: An example

$$\varphi_{\mathrm{PM}}(t) = A_c \cos\big(2\pi f_c t + K_p m(t) + \theta_0\big)$$

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = \begin{cases} f_c + \frac{d}{dt}m(t), & \text{if } \textbf{PM} \end{cases}$$



Angle Modulation: An example

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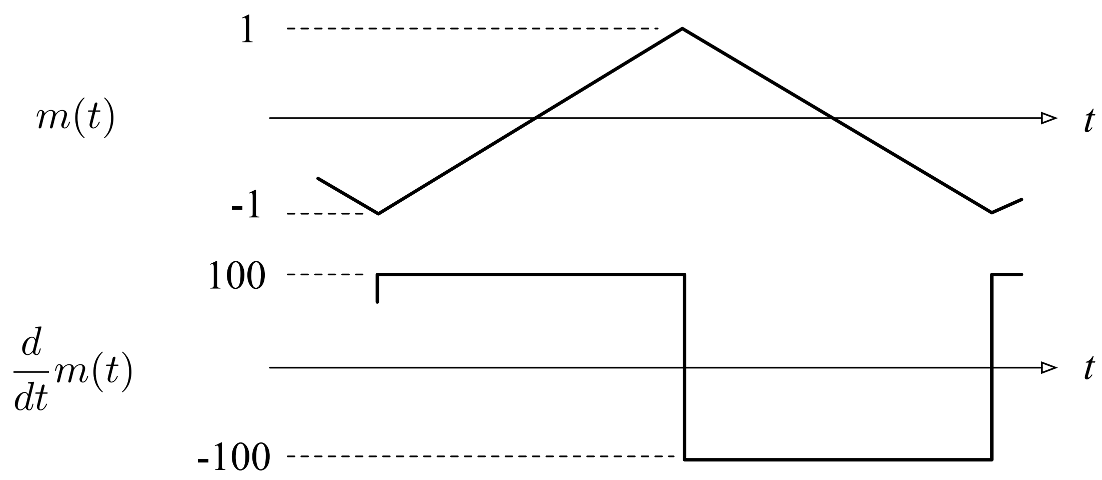

Angle Modulation: An example

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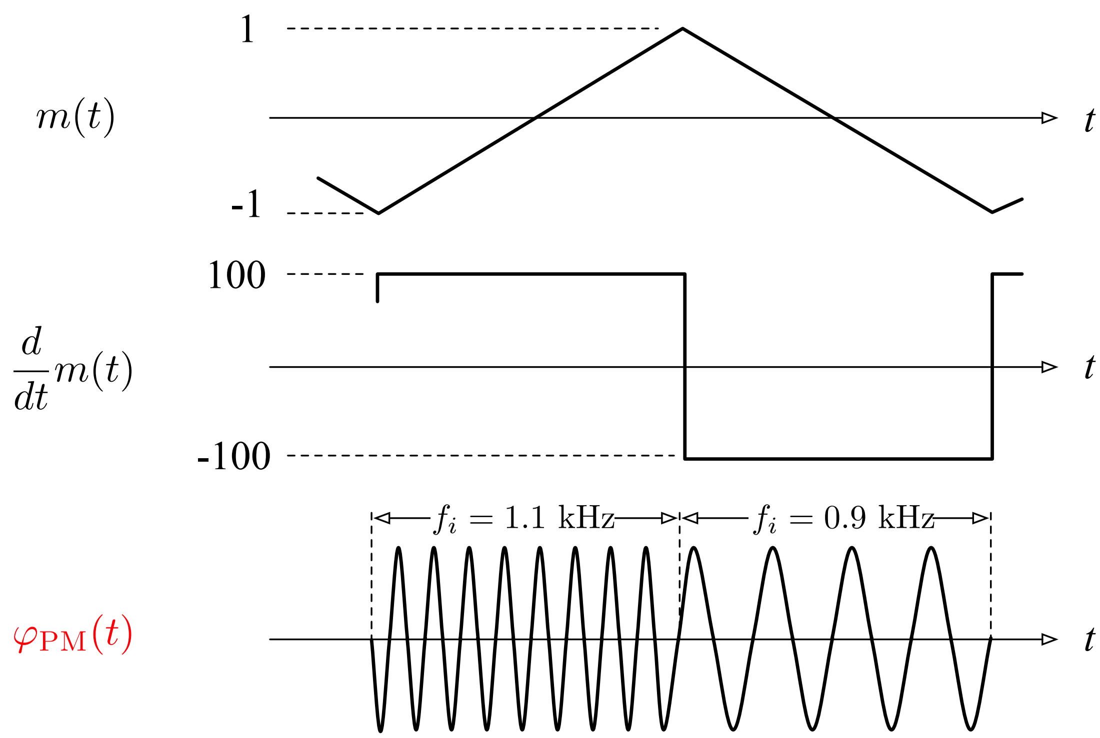
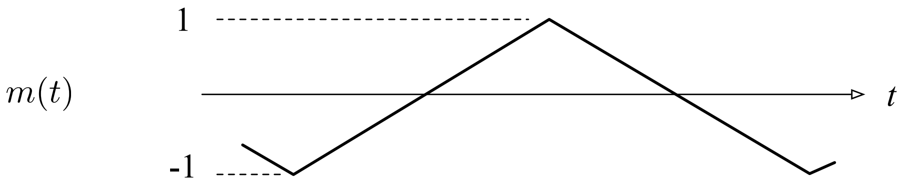


Angle Modulation: An example

$$\varphi_{\mathrm{FM}}(t) = A_c \cos\left(2\pi f_c t + K_f \int_0^t m(\lambda) d\lambda + \theta_0\right)$$

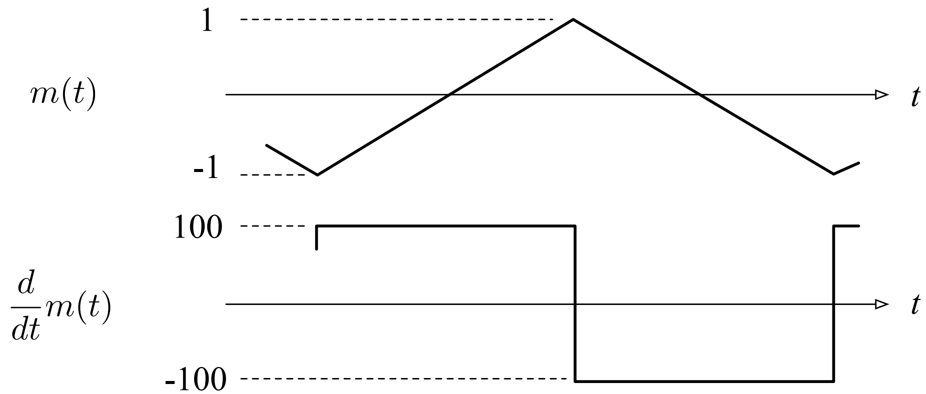
$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = \begin{cases} \\ f_c + m(t), & \text{if FM} \end{cases}$$



Angle Modulation: An example

$$\varphi_{\rm FM}(t) = A_c \cos\left(2\pi f_c t + K_f \int_0^t m(\lambda) d\lambda + \theta_0\right)$$

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = \begin{cases} \\ f_c + m(t), & \text{if } \textbf{FM} \end{cases}$$

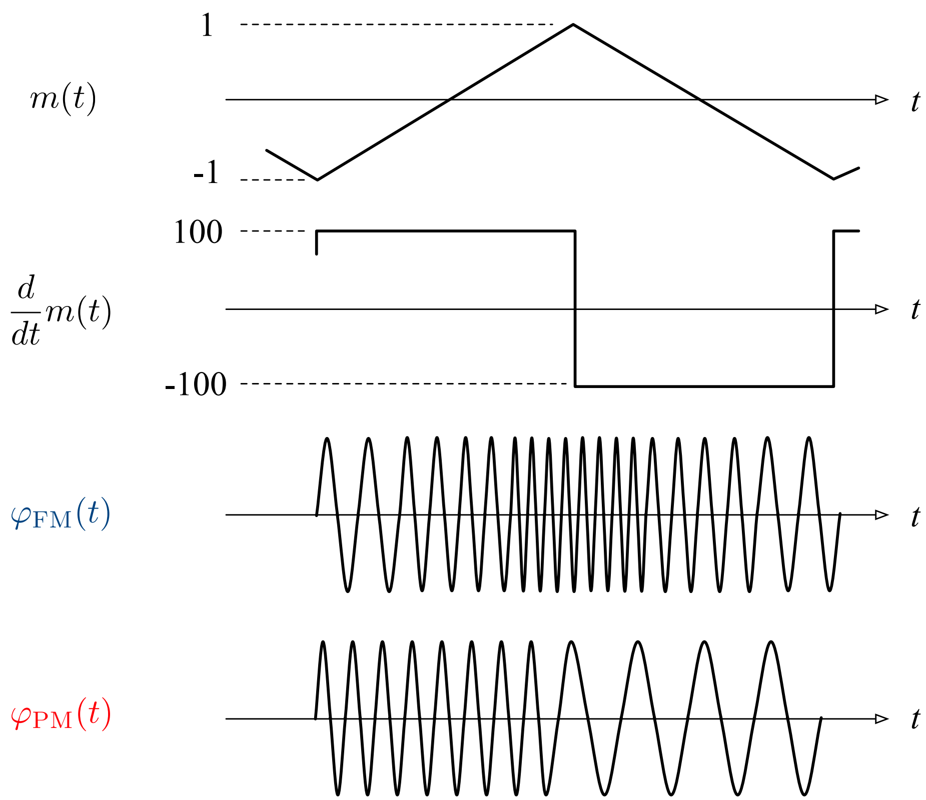


Angle Modulation: An example

$$\varphi_{\mathrm{PM}}(t) = A_c \cos\left(2\pi f_c t + K_p m(t) + \theta_0\right)$$

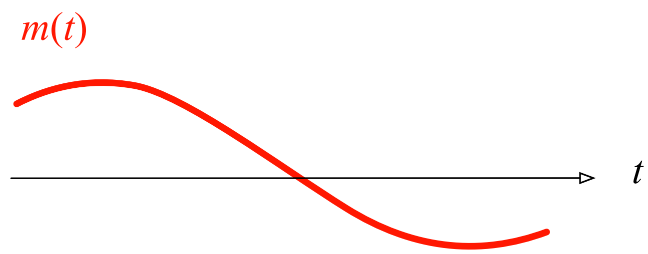
$$\varphi_{\mathrm{FM}}(t) = A_c \cos\left(2\pi f_c t + K_f \int_0^t m(\lambda) d\lambda + \theta_0\right)$$

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = \begin{cases} f_c + \frac{d}{dt}m(t), & \text{if PM} \\ f_c + m(t), & \text{if FM} \end{cases}$$



FM: Bandwidth estimation

Consider the modulating waveform $m(t)$



FM: Bandwidth estimation

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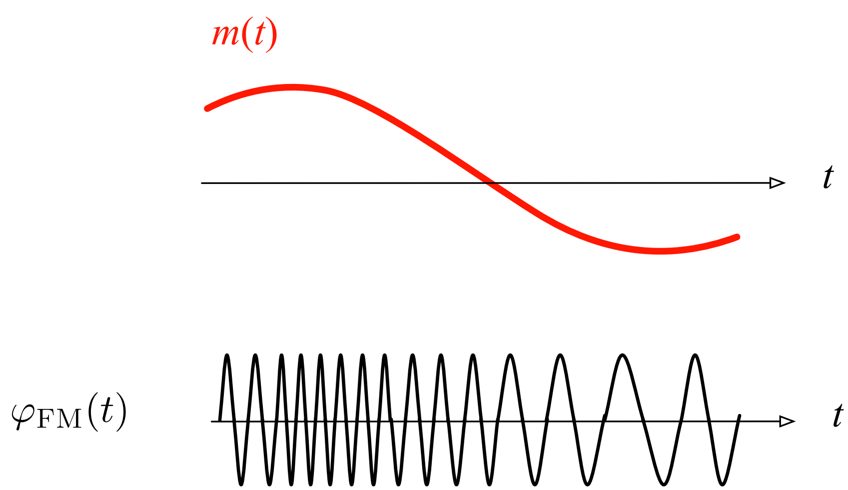


Objective: Determine $\Phi_{FM}(f)$, spectrum of the FM signal $\varphi_{FM}(t)$ generated by an arbitrary modulating signal $m(t)$.

No easy task! … as FM is a non-linear operation ⟶ **approximation**

FM: Bandwidth estimation

Let's start with the simple case of a single-tone modulation:

$$m(t) = A_m \cos \omega_m t$$

such that:

$$\varphi_{\mathrm{FM}}(t) = A_c \cos\left(\omega_c t + K_f \int_0^t m(\lambda) d\lambda\right)$$

with the instantaneous frequency of the FM signal given as:

$$f_i(t) = f_c + \frac{K_f}{2\pi} m(t)$$

$$= f_c + \frac{K_f A_m}{2\pi} \cos \omega_m t$$

$$= f_c + \Delta f \cos \omega_m t$$

FM: Bandwidth estimation

Observe that:

$$\Delta f = \frac{K_f A_m}{2\pi}$$

represents the **maximum frequency deviation** of the instantaneous frequency $f_i(t)$ from the unmodulated carrier frequency f_c

$$\Delta f \sim A_m = \max_t \left[m(t) \right]$$

The maximum frequency deviation Δf is independent of f_m but rather is a function of the maximum signal amplitude.

FM: Bandwidth estimation

Let us now rewrite the FM signal $\varphi_{\text{FM}}(t)$ by computing the time-varying phase:

$$
\begin{aligned}
\theta(t) &= 2\pi \int_0^t f_i(\lambda) d\lambda \\
&= 2\pi \int_0^t \left[f_c + \frac{K_f A_m}{2\pi} \cos \omega_m \lambda \right] d\lambda \\
&= 2\pi \left[f_c \lambda + \frac{K_f A_m}{2\pi} \frac{\sin \omega_m \lambda}{\omega_m} \Big|_0^t \right] \\
&= 2\pi f_c t + \frac{K_f A_m}{2\pi f_m} \sin \omega_m t \\
&= 2\pi f_c t + \frac{\Delta f}{f_m} \sin \omega_m t
\end{aligned}
$$

FM: Bandwidth estimation

Using the simplified notation:

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta \omega}{\omega_m} = \frac{K_f A_m / 2\pi}{f_m}$$

we express the FM signal as:

$$\begin{aligned}
\varphi_{\mathrm{FM}}(t) &= A_c \cos\theta(t) \\
&= A_c \cos\left(2\pi f_c t + \frac{\Delta f}{f_m} \sin \omega_m t\right) \\
&= A_c \cos\left(2\pi f_c t + \beta \sin \omega_m t\right) \\
&= A_c \; \cos\omega_c t \; \cos(\beta \sin\omega_m t) - A_c \; \sin\omega_c t \; \sin(\beta \sin\omega_m t)
\end{aligned}$$

FM: Bandwidth estimation

$$\varphi_{\text{FM}}(t) = A_c \cos\omega_c t \cos\big(\beta \sin\omega_m t\big) - A_c \sin\omega_c t \sin\big(\beta \sin\omega_m t\big)$$

Case 1 Narrowband FM with β small (typically $\beta < 0.3$)

$$\cos\big(\beta \sin\omega_m t\big) \approx 1 \qquad \text{and} \qquad \sin\big(\beta \sin\omega_m t\big) \approx \beta \sin\omega_m t$$

Therefore:

$$\varphi_{\text{FM}}(t) \approx A_c \cos\omega_c t - A_c \beta \sin\omega_c t \sin\omega_m t$$

FM: Bandwidth estimation

How does the narrowband FM (NBFM) case with $m(t) = A_m \cos \omega_m t$ compare with the single-tone AM signal?

$$\begin{aligned}
\varphi_{\mathrm{AM}}(t) &= \left[A_c + A_m \cos \omega_m t \right] \cos \omega_c t \\
&= A_c \left[1 + \frac{A_m}{A_c} \cos \omega_m t \right] \cos \omega_c t \\
&= A_c \left[1 + \mu \cos \omega_m t \right] \cos \omega_c t \\
&= A_c \cos \omega_c t + A_c \, \mu \, \cos \omega_m t \cos \omega_c t
\end{aligned}$$

FM: Bandwidth estimation

How does the narrowband FM (NBFM) case with $m(t) = A_m \cos \omega_m t$ compare with the single-tone AM signal?

$$\varphi_{\mathrm{AM}}(t) = \left[\, A_c + A_m \cos \omega_m t \,\right] \cos \omega_c t$$

$$= A_c \left[\, 1 + \frac{A_m}{A_c} \cos \omega_m t \,\right] \cos \omega_c t$$

$$= A_c \left[\, 1 + \mu \cos \omega_m t \,\right] \cos \omega_c t$$

$$= A_c \cos \omega_c t + A_c \, \mu \, \cos \omega_m t \cos \omega_c t$$

$$\varphi_{\mathrm{AM}}(t) = A_c \cos \omega_c t + A_c \, \mu \, \cos \omega_m t \cos \omega_c t$$

$$\varphi_{\mathrm{NBFM}}(t) = A_c \cos \omega_c t - A_c \, \beta \, \sin \omega_m t \sin \omega_c t$$

FM: Bandwidth estimation

$$\varphi_{\mathrm{AM}}(t) = A_c \cos \omega_c t + A_c \, \mu \, \cos \omega_m t \cos \omega_c t$$

$\Updownarrow$

$$\varphi_{\mathrm{NBFM}}(t) = A_c \cos \omega_c t - A_c \, \beta \, \sin \omega_m t \sin \omega_c t$$

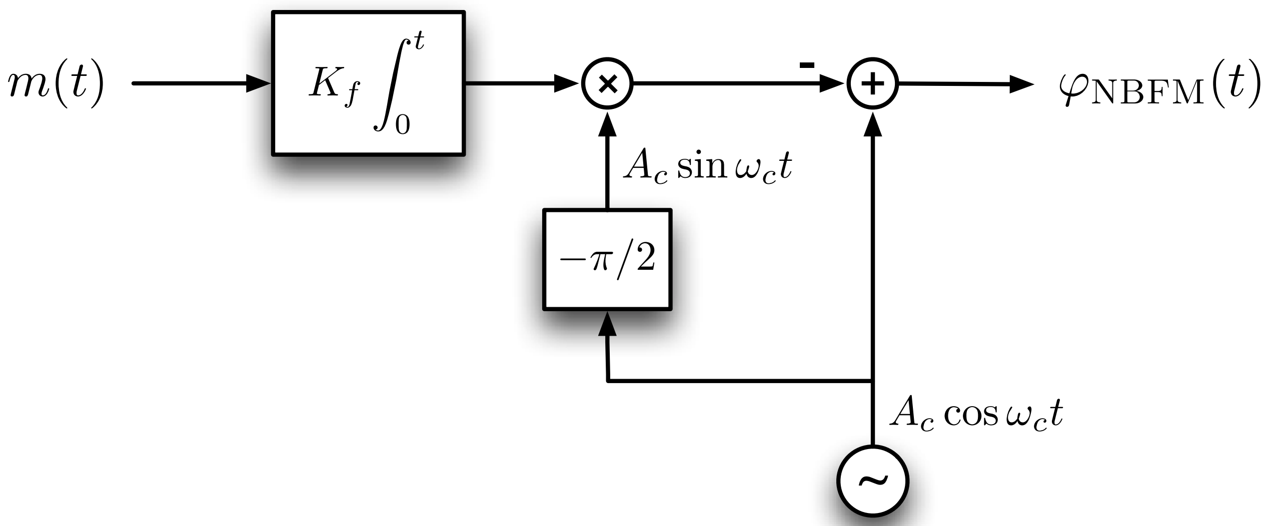
Observations

- Single-tone AM and NBFM are similar, yet they are distinct modulation schemes.
- In view of the similarity between the AM modulation index μ and $\beta = (\Delta f \,/\, f_m)$ we will refer to β as the **modulation index** for the FM signals (applicable only for single-tone modulation).

FM: Bandwidth estimation

$$\varphi_{\text{NBFM}}(t) = A_c \cos\omega_c t - A_c\,\beta\,\sin\omega_m t \sin\omega_c t$$

Generating NBFM signals

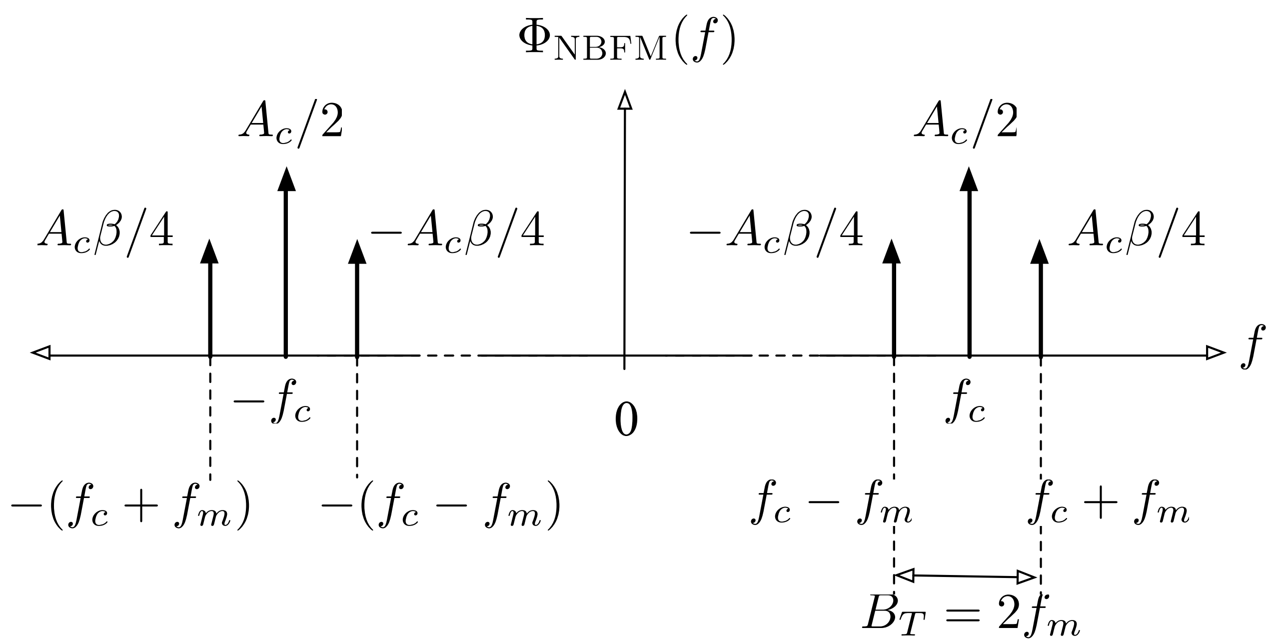


FM: Bandwidth estimation

Spectrum of a single-tone modulated NBFM signal

$$\varphi_{\mathrm{NBFM}}(t) = A_c \cos \omega_c t - A_c\, \beta\, \sin \omega_m t \sin \omega_c t$$

$$= A_c \cos \omega_c t - \frac{A_c \beta}{2} \left[\cos(\omega_c - \omega_m)t - \cos(\omega_c + \omega_m)t \right]$$

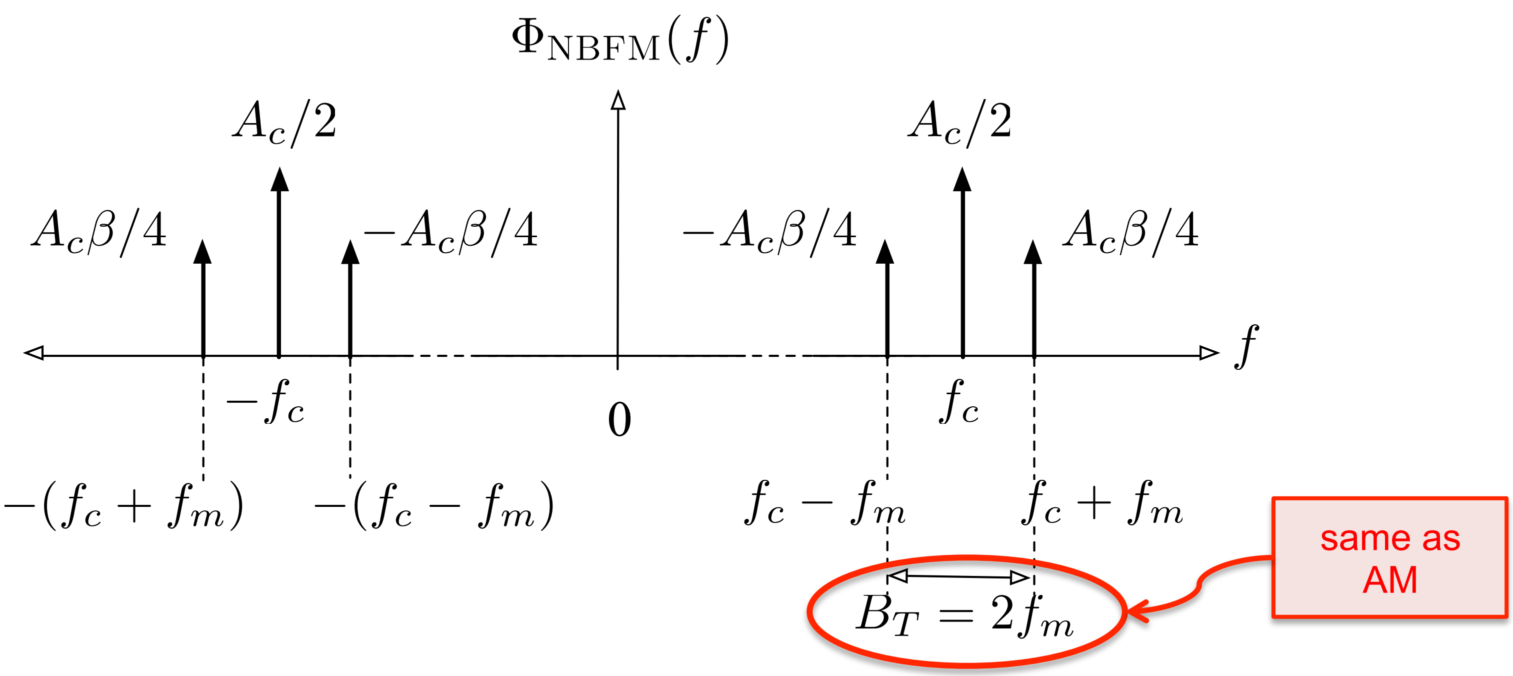


FM: Bandwidth estimation

Spectrum of a single-tone modulated NBFM signal

$$\varphi_{\text{NBFM}}(t) = A_c \cos \omega_c t - A_c\, \beta\, \sin \omega_m t \sin \omega_c t$$

$$= \boxed{A_c \cos \omega_c t - \frac{A_c \beta}{2} \left[\cos(\omega_c - \omega_m)t - \cos(\omega_c + \omega_m)t \right]}$$



FM: Bandwidth estimation

So far all what we have achieved was to determine the spectrum of a single-tone modulated NBFM signal ... As we stated at the onset **computing the spectrum for the more general cases is no easy task !**

FM: Bandwidth estimation

So far all what we have achieved was to determine the spectrum of a single-tone modulated NBFM signal ... As we stated at the onset **computing the spectrum for the more general cases is no easy task !**

Let us consider a more general case with an arbitrary $m(t)$ with:

$$a(t) = \int_0^t m(\lambda)\,\lambda$$

and express the FM signal in a form using the simplified notation:

$$\varphi_{\text{FM}}(t) = A_c \cos\big(\,\omega_c t + K_f \int_0^t m(\lambda) d\lambda\,\big)$$

$$= A_c \cos\big(\,\omega_c t + K_f a(t)\,\big)$$

FM: Bandwidth estimation

We can also express $\varphi_{\text{FM}}(t)$ as:

$$\begin{aligned}
\varphi_{\text{FM}}(t) &= A_c \cos\Big(\omega_c t + K_f \int_0^t m(\lambda)d\lambda\Big) \\
&= A_c \cos\big(\omega_c t + K_f a(t)\big) \\
&= \mathbf{Re}\Big\{A_c e^{j[\omega_c t + K_f a(t)]}\Big\} \\
&= \mathbf{Re}\Big\{A_c\left[1 + jK_f a(t) - K_f^2 \frac{a^2(t)}{2!} + \cdots\right] e^{j\omega_c t}\Big\} \\
&= A_c\Big[\cos\omega_c t - K_f a(t)\sin\omega_c t - \frac{K_f^2}{2!} a^2(t)\cos\omega_c t + \cdots\Big]
\end{aligned}$$

FM: Bandwidth estimation

Assume that $m(t)$ is bandlimited to B_m Hz. Then:

$$\varphi_{\mathrm{FM}}(t) = A_c\left[\cos\omega_c t - K_f a(t)\sin\omega_c t - \frac{K_f^2}{2!}a^2(t)\cos\omega_c t - \frac{K_f^3}{3!}a^3(t)\sin\omega_c t + \cdots\right]$$

$a(t)$ is bandlimited to B_m Hz

FM: Bandwidth estimation

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$a(t)$ is bandlimited to B_m Hz

$a^2(t)$ is bandlimited to $2B_m$ Hz

FM: Bandwidth estimation

Assume that $m(t)$ is bandlimited to B_m Hz. Then:

$$\varphi_{\mathrm{FM}}(t) = A_c\left[\cos\omega_c t - K_f a(t)\sin\omega_c t - \frac{K_f^2}{2!}a^2(t)\cos\omega_c t - \frac{K_f^3}{3!}a^3(t)\sin\omega_c t + \cdots\right]$$

$a(t)$ is bandlimited to B_m Hz

$a^2(t)$ is bandlimited to $2B_m$ Hz

$a^3(t)$ is bandlimited to $3B_m$ Hz

....

$a^n(t)$ is bandlimited to nB_m Hz

FM: Bandwidth estimation

Assume that $m(t)$ is bandlimited to B_m Hz. Then:

$$\varphi_{\mathrm{FM}}(t) = A_c\left[\cos\omega_c t - K_f a(t)\sin\omega_c t - \frac{K_f^2}{2!}a^2(t)\cos\omega_c t - \frac{K_f^3}{3!}a^3(t)\sin\omega_c t + \cdots\right]$$

$a(t)$ is bandlimited to B_m Hz

$a^2(t)$ is bandlimited to $2B_m$ Hz

$a^3(t)$ is bandlimited to $3B_m$ Hz

.....

$a^n(t)$ is bandlimited to nB_m Hz

The FM signal $\varphi_{\mathrm{FM}}(t)$ is **not bandlimited !** ...

FM: Bandwidth estimation

What about the NBFM case? NBFM (with single-tone modulation)

$$\beta = \frac{\Delta f}{f_m} = \frac{K_f A_m}{2\pi f_m}$$ is small $\Rightarrow$ K_f is small

FM: Bandwidth estimation

What about the NBFM case? NBFM (with single-tone modulation)

$$\beta = \frac{\Delta f}{f_m} = \frac{K_f A_m}{2\pi f_m}$$ is small ⇒ K_f is small

such that

$$\varphi_{\mathrm{FM}}(t) = A_c\left[\cos\omega_c t - K_f a(t)\sin\omega_c t - \frac{K_f^2}{2!}a^2(t)\cos\omega_c t - \frac{K_f^3}{3!}a^3(t)\sin\omega_c t + \cdots\right]$$

$$\approx A_c\cos\omega_c t - A_c K_f a(t)\sin\omega_c t$$

as $K_f \gg K_f^2 \gg K_f^3 \gg \cdots$

FM: Bandwidth estimation

Given that

$$\varphi_{\mathrm{FM}}(t) \approx A_c \cos\omega_c t - A_c K_f a(t) \sin\omega_c t$$

B_{T} transmission bandwidth of the NBFM signal $\varphi_{\mathrm{FM}}(t)$

$$\begin{aligned} B_{\mathrm{T}} &= 2 \text{ Bandwidth of } [\ a(t)\ ] \\ &= 2 \text{ Bandwidth of } [\ m(t)\ ] \\ &= 2B_m \end{aligned}$$

which is what we determined earlier.

FM: Bandwidth estimation

Given that

$$\varphi_{\mathrm{FM}}(t) \approx A_c \cos \omega_c t - A_c K_f a(t) \sin \omega_c t$$

B_{T} transmission bandwidth of the NBFM signal $\varphi_{\mathrm{FM}}(t)$

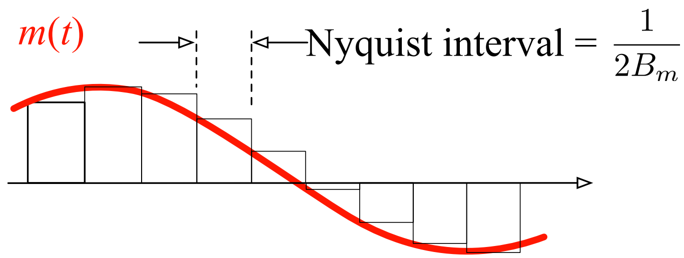
$$\begin{aligned} B_{\mathrm{T}} &= 2 \text{ Bandwidth of } [\ a(t)\ ] \\ &= 2 \text{ Bandwidth of } [\ m(t)\ ] \\ &= 2\, B_m \end{aligned}$$

which is what we determined earlier.

As we will show shortly the result that for NBFM signals $B_{\mathrm{T}} = 2B_m$ will be valid for all NBFM signals generated by bandlimited modulating signals.

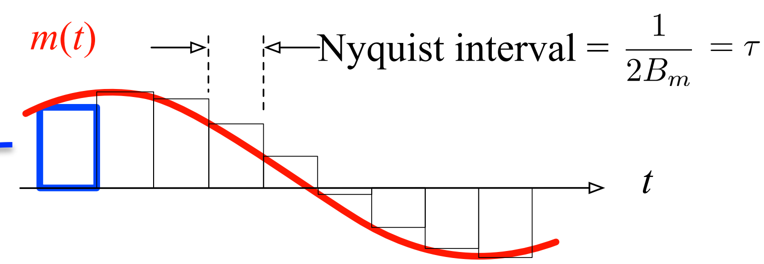
FM: Bandwidth estimation

Consider the modulating waveform $m(t)$

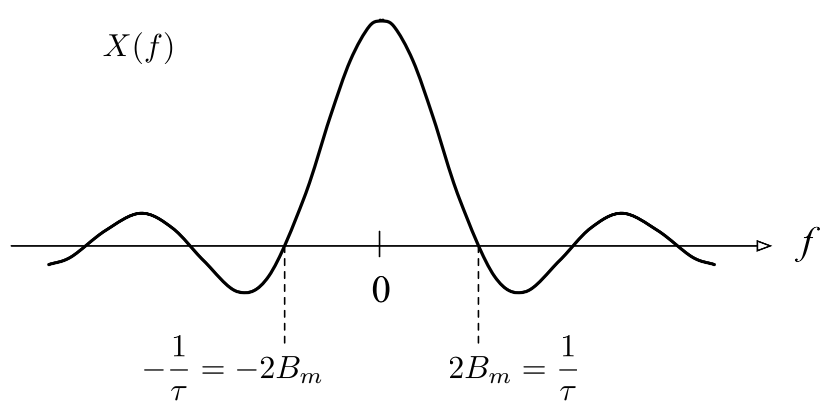


FM: Bandwidth estimation

Consider the modulating waveform $m(t)$



$$x(t) = \Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow X(f) = \tau \mathrm{sinc}(\pi f \tau)$$



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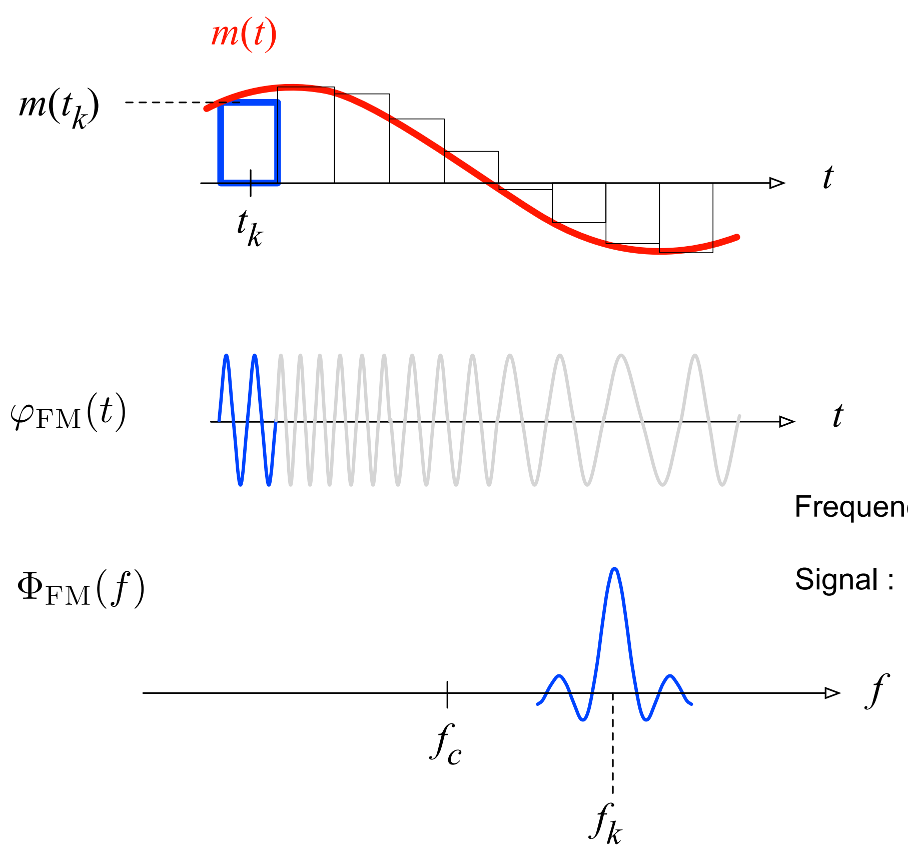
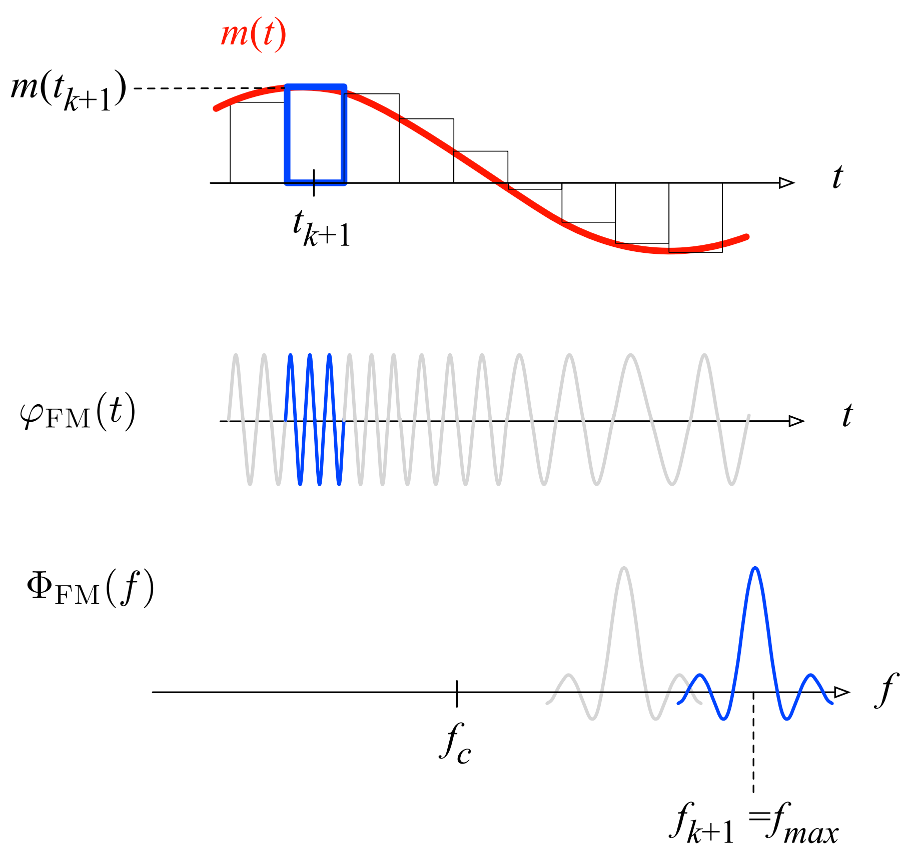


Frequency : $f_i = f_c + \frac{K_f}{2\pi} m(t_k) = f_k$

Signal : $\cos(2\pi f_k t)\,\Pi\big(2B_m(t - t_k)\big)$

FM: Bandwidth estimation

Consider the modulating waveform $m(t)$



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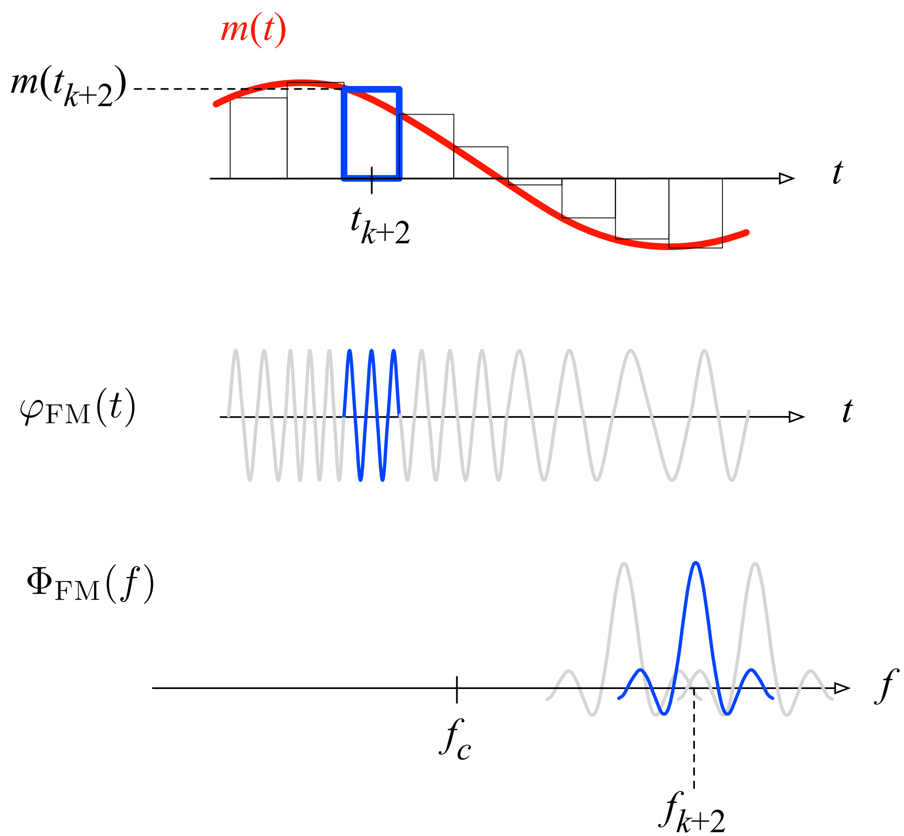

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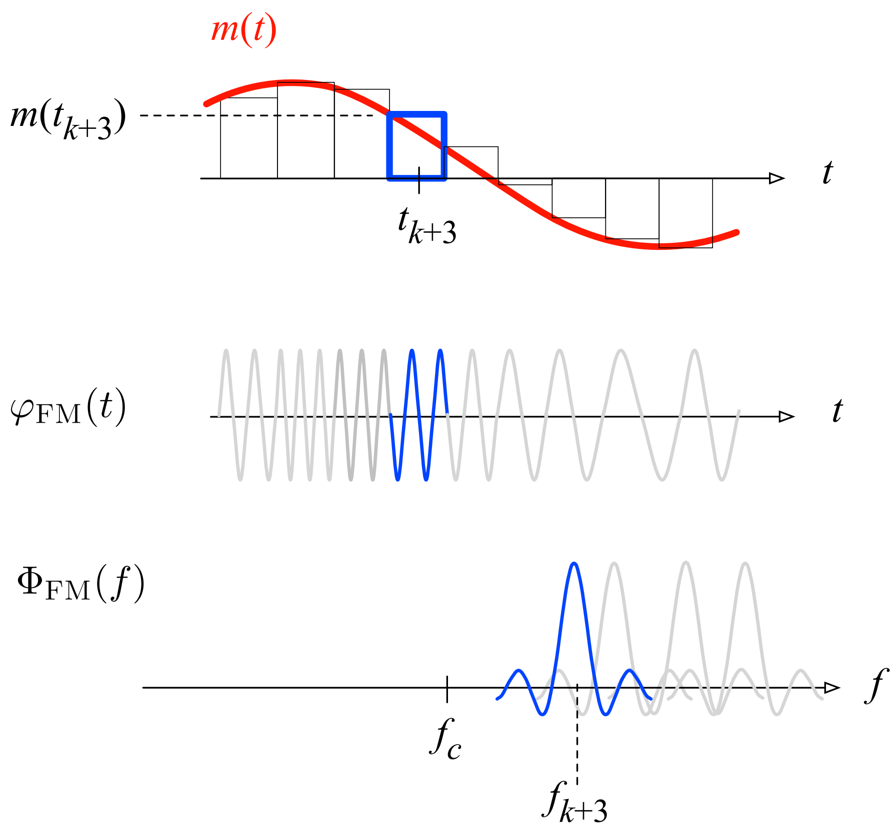

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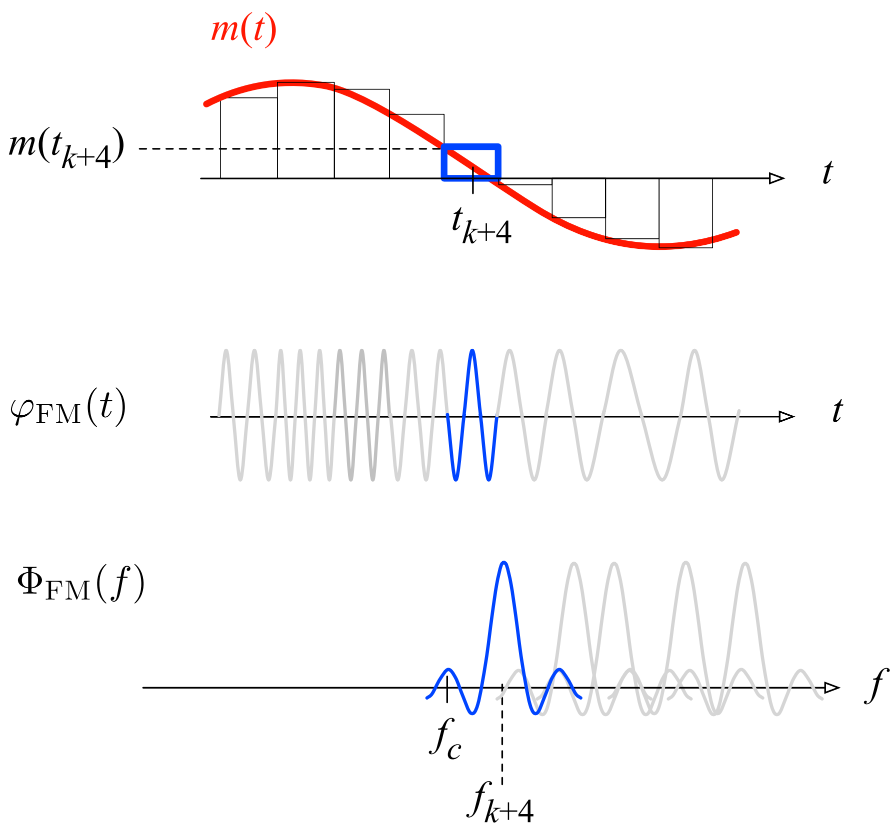

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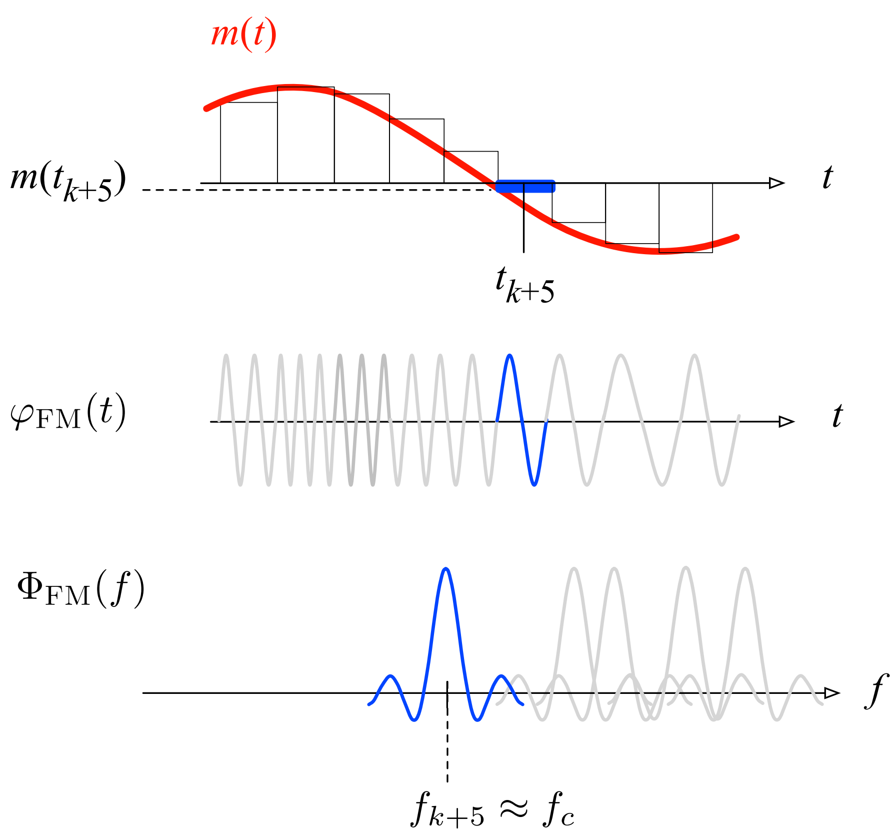

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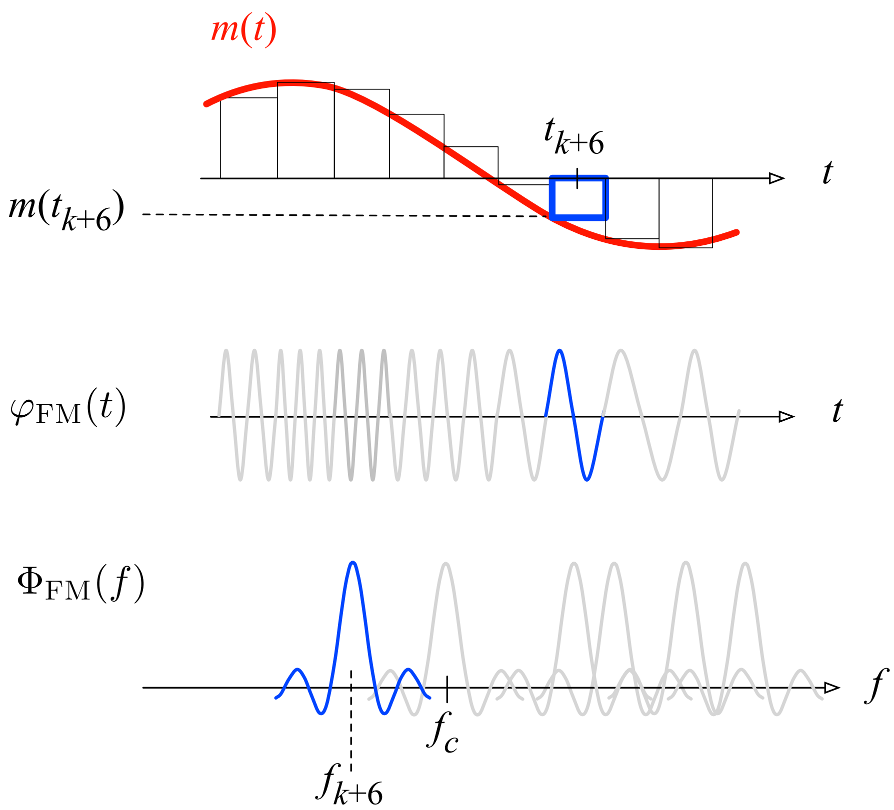

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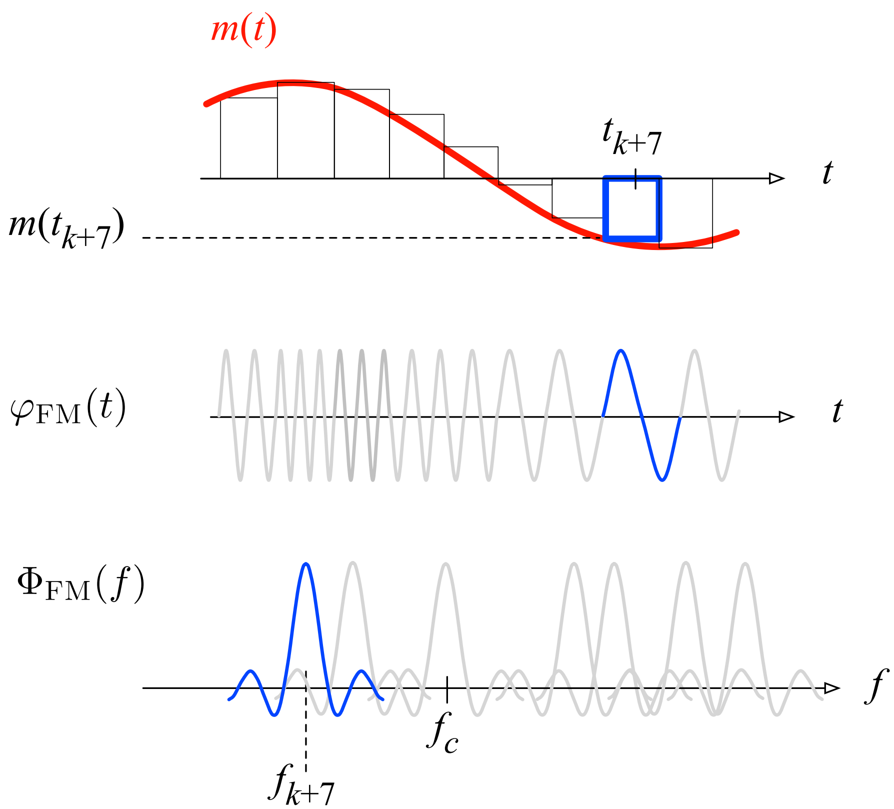

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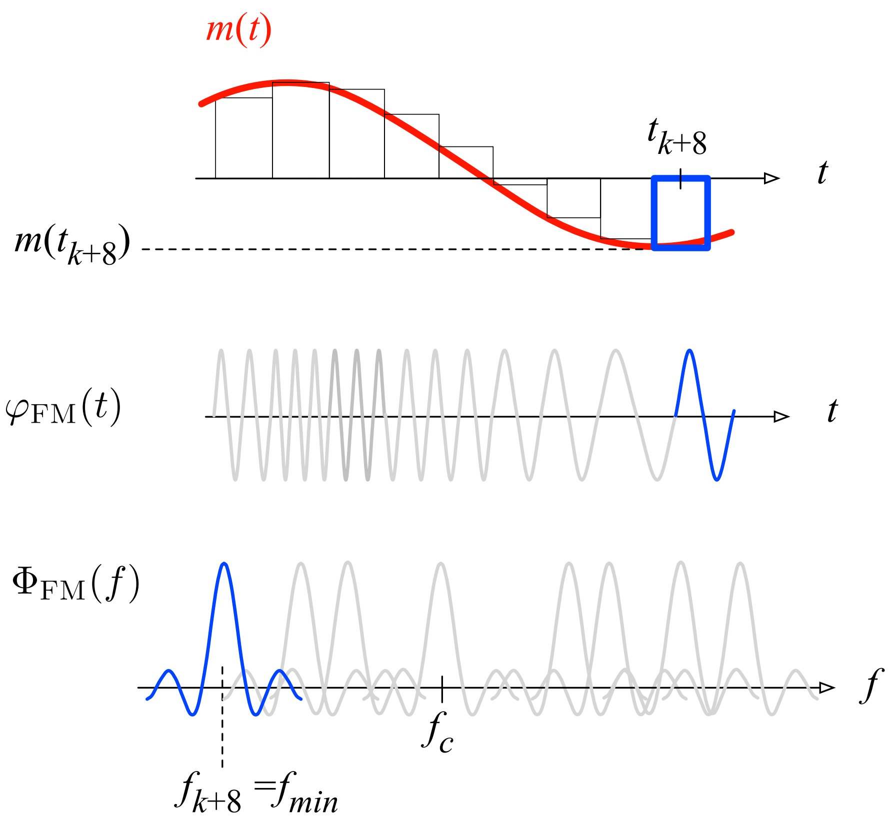

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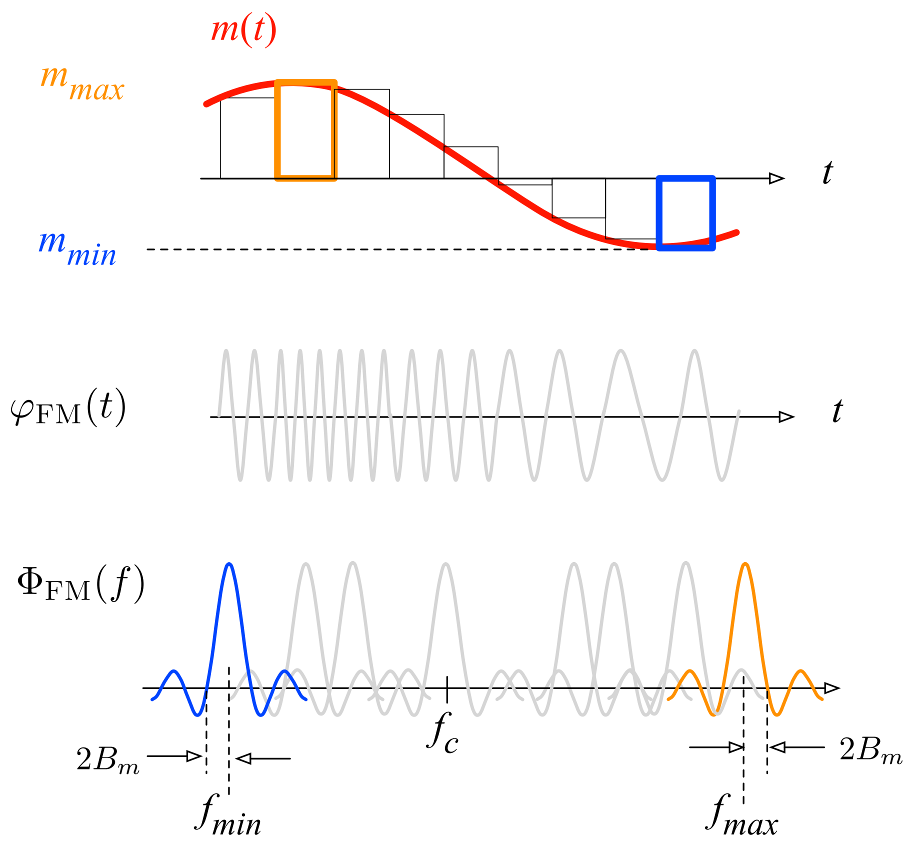

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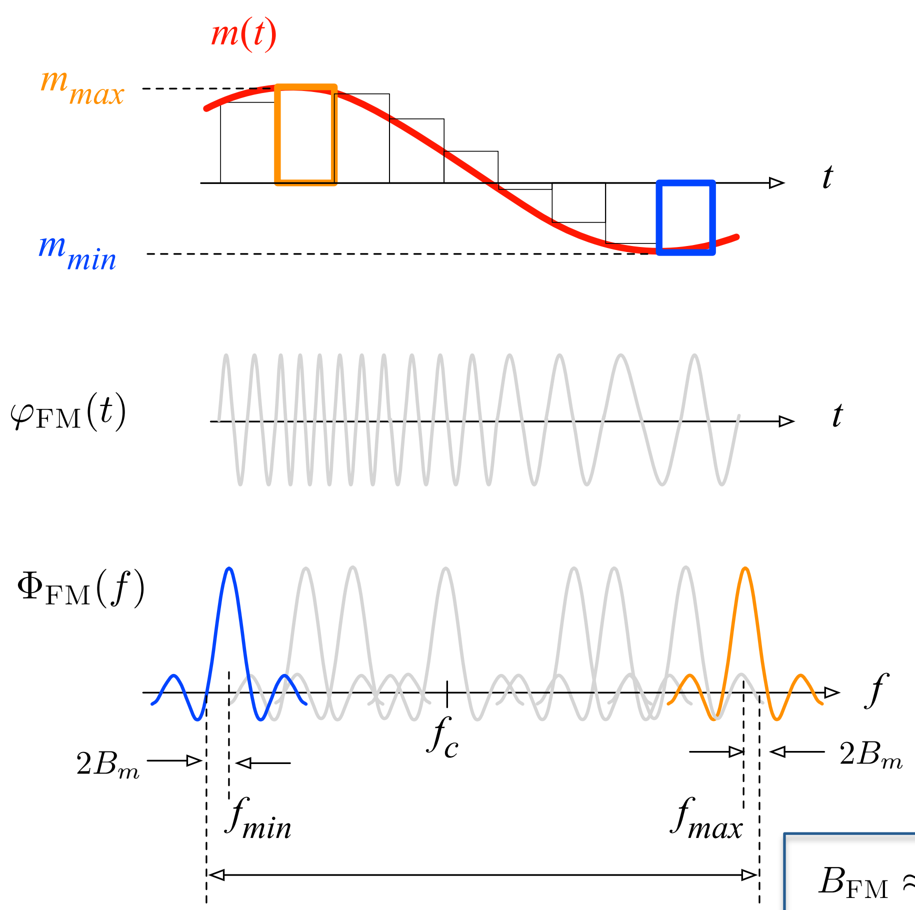
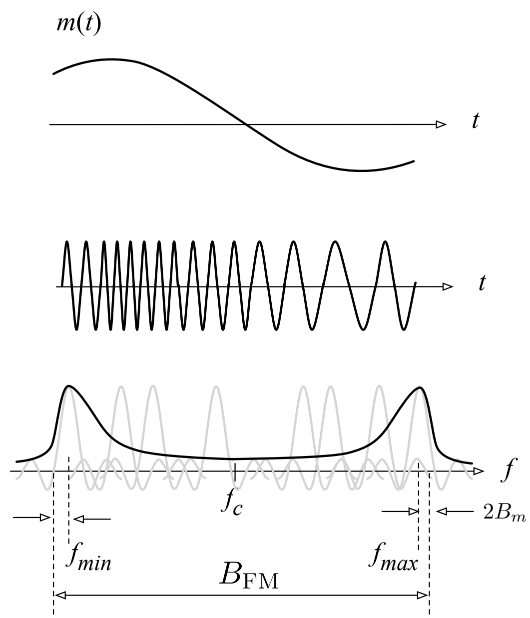


$$B_{\text{FM}} \approx [f_{max} - f_{min}] + 2(2B_m)$$

FM: Bandwidth estimation



$$f_{min} = f_c + \frac{K_f}{2\pi}\min_t[m(t)] = f_c - \frac{K_f}{2\pi}m_p$$

$$f_{max} = f_c + \frac{K_f}{2\pi}\max_t[m(t)] = f_c + \frac{K_f}{2\pi}m_p$$

$$\begin{aligned} B_{\text{FM}} &\approx \left[f_{max} - f_{min}\right] + 2(2B_m) \\ &= 2\frac{K_f}{2\pi}m_p + 2(2B_m) \\ &= 2(\Delta f + 2B_m) \end{aligned}$$

FM: Bandwidth estimation

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\begin{aligned}
B_{\mathrm{FM}} &\approx \left[f_{max} - f_{min}\right] + 2(2B_m) \\
&= 2\frac{K_f}{2\pi} m_p + 2(2B_m) \\
&= 2(\Delta f + 2B_m)
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$$

- This approximation to B_{FM} is not a very good one:
  - → in the **narrowband FM** case $\Delta f \approx 0$ so that $B_{\mathrm{NBFM}} \approx 2B_m$.

FM: Bandwidth estimation

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- This approximation to B_{FM} is not a very good one:
  → in the **narrowband FM** case $\Delta f \approx 0$ so that $B_{\mathrm{NBFM}} \approx 2B_m$.

- A better approximation is given by **Carson's Rule**:

$$B_{\mathrm{FM}} \approx 2(\Delta f + B_m)$$

FM: Bandwidth estimation − Carson's Rule

$$B_{\text{FM}} \approx 2(\Delta f + B_m)$$

Define:

Modulation Index: $$\beta = \frac{\Delta f}{B_m} = \frac{\Delta f}{f_m}$$

Deviation Index: $$\mathcal{D} = \frac{\Delta f}{B_m}$$

FM: Bandwidth estimation – Carson's Rule

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For **single-tone** modulation only !!!

FM: Bandwidth estimation − Carson's Rule

$$B_{\text{FM}} \approx 2(\Delta f + B_m)$$

Single Tone Modulation:

$$\beta = \frac{\Delta f}{f_m} \quad \Longrightarrow \quad B_{\text{FM}} \approx 2B_m(1 + \beta)$$

Wideband Signal Modulation:

$$\mathcal{D} = \frac{\Delta f}{B_m} \quad \Longrightarrow \quad B_{\text{FM}} \approx 2B_m(1 + \mathcal{D})$$

FM: Bandwidth estimation

Determining a closed-form expression for the spectrum of signal $\varphi_{\mathrm{FM}}(t)$ for arbitrary modulation/deviation index values is not possible. Therefore we will investigate the single-tone modulation case:

$$m(t) = A_m \cos \omega_m t$$

such that:

$$\begin{aligned}
\varphi_{\mathrm{FM}}(t) &= A_c \cos\Big(\omega_c t + K_f \int_0^t m(\lambda) d\lambda\Big) \\
&= A_c \cos\Big(2\pi f_c t + \beta \sin \omega_m t\Big) \\
&= \mathbf{Re}\Big\{A_c \, e^{\,j[\omega_c t + \beta \sin \omega_m t]}\Big\} \\
&= \mathbf{Re}\Big\{A_c \, e^{\,j\beta \sin \omega_m t} \, e^{\,j\omega_c t}\Big\}
\end{aligned}$$

FM: Bandwidth estimation

$$\varphi_{\mathrm{FM}}(t) = \mathbf{Re}\left\{A_c\, e^{j\beta \sin \omega_m t}\, e^{j\omega_c t}\right\}$$

The complex exponential function $e^{j\beta \sin \omega_m t}$ is periodic, and therefore it can be expanded in the Fourier series:

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}$$

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$$C_n = \frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta \sin \omega_m t}\, e^{-jn\omega_m t} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx, \quad \text{with} \quad x = \omega_m t$$

$$= J_n(\beta)$$

FM: Bandwidth estimation

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What is this $J_n(\beta)$?

FM: Bandwidth estimation

$$J_n(\beta)$$

J : Bessel function of 1st kind.

n : order

β : argument

Bessel function of 1st kind "J", order n and argument β

FM: Bandwidth estimation

$$J_n(\beta)$$

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Bessel function of 1st kind (J), order n and argument β

Shortly, we will return to discuss the properties of Bessel functions and methods of evaluating them. But for the time being assume that $J_n(\beta)$ are readily available so that we can proceed with evaluating the spectrum of the FM signal.

FM: Bandwidth estimation

$$
\begin{aligned}
\varphi_{\mathrm{FM}}(t) &= \mathbf{Re}\left\{ A_c\, e^{j\beta \sin \omega_m t}\, e^{j\omega_c t} \right\} \\
&= \mathbf{Re}\left\{ A_c \left[\sum_n J_n(\beta) e^{jn\omega_m t} \right] e^{j\omega_c t} \right\} \\
&= \mathbf{Re}\left\{ A_c \sum_n J_n(\beta) e^{j(\omega_c + n\omega_m)t} \right\} \\
&= \boxed{A_c \sum_n J_n(\beta) \cos(\omega_c + n\omega_m)t}
\end{aligned}
$$

The last expression is in a form suitable for computing the spectrum of the FM signal.

FM: Bandwidth estimation

$$\varphi_{\mathrm{FM}}(t) = A_c \sum_n J_n(\beta)\cos(\omega_c + n\omega_m)t$$

such that:

$$\begin{aligned}
\Phi_{\mathrm{FM}}(f) &= \mathcal{F}\left[\varphi_{\mathrm{FM}}(t)\right] \\
&= \mathcal{F}\left[A_c \sum_n J_n(\beta)\cos(\omega_c + n\omega_m)t\right] \\
&= A_c \sum_n J_n(\beta)\,\mathcal{F}\left[\cos(\omega_c + n\omega_m)t\right] \\
&= \frac{A_c}{2} \sum_n J_n(\beta)\left[\delta\big(f - (f_c + nf_m)\big) + \delta\big(f + (f_c + nf_m)\big)\right]
\end{aligned}$$

FM: Bandwidth estimation

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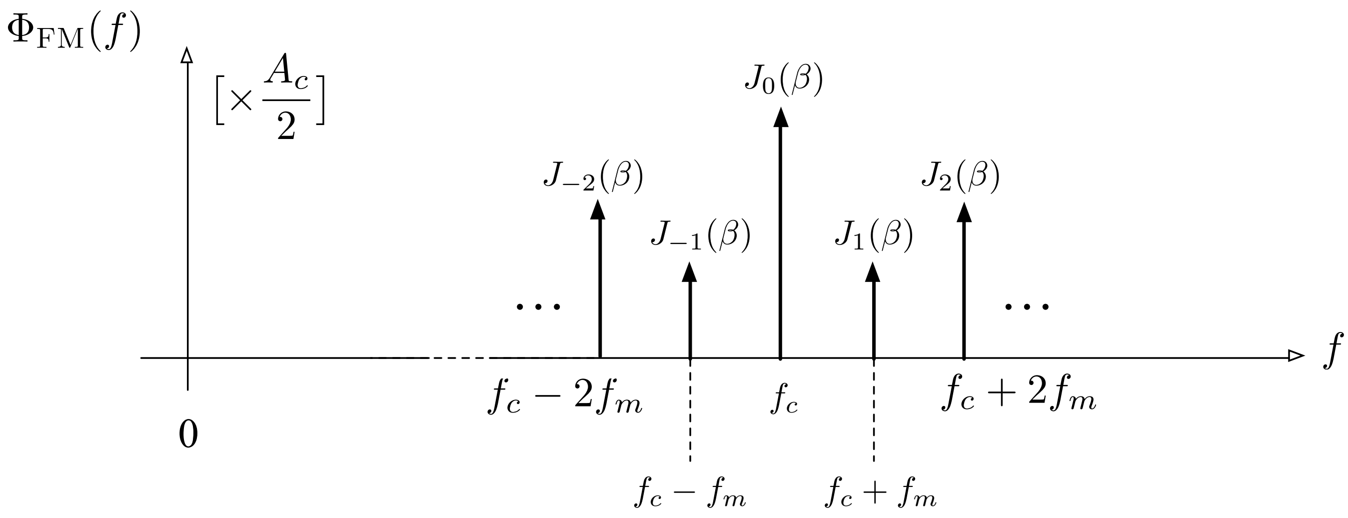

FM: Bandwidth estimation

$$\Phi_{\mathrm{FM}}(f) = \frac{A_c}{2}\sum_n J_n(\beta)\left[\delta\big(f-(f_c+nf_m)\big)+\delta\big(f+(f_c+nf_m)\big)\right]$$

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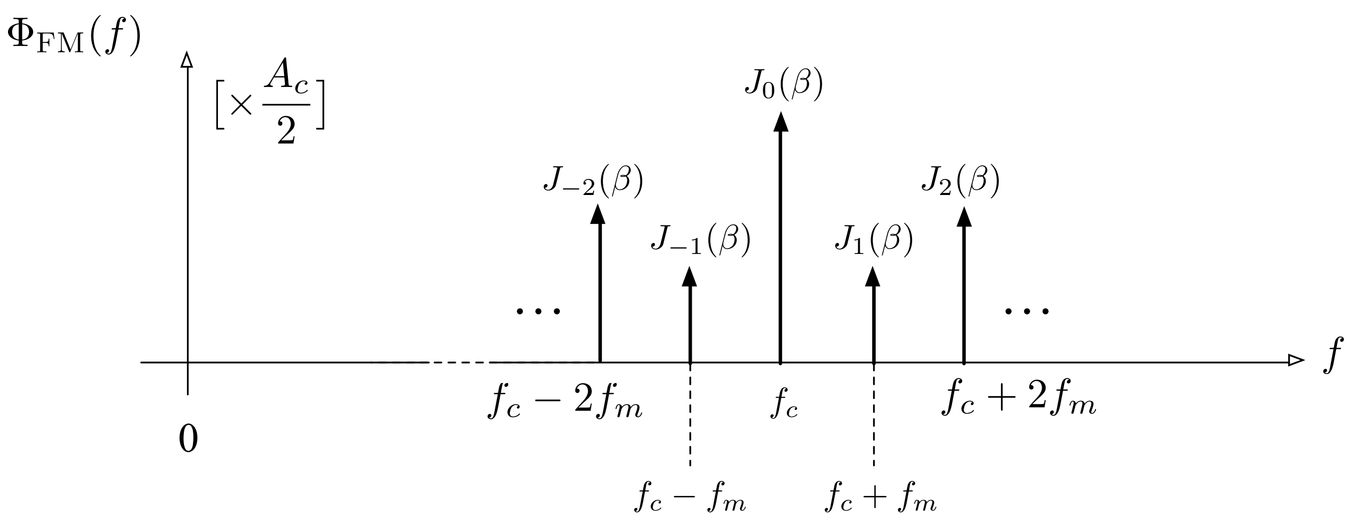

- There are an **infinite number of sidebands** → not bandlimited.
- β and in turn $J_n(\beta)$ values determine the shape of $\Phi_{\mathrm{FM}}(f)$.

Properties of Bessel functions

- $J_n(\beta)$ is **real-valued**.

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$$J_0(\beta) \approx 1 \qquad J_1(\beta) = -J_{-1}(\beta) = \frac{\beta}{2} \qquad J_n(\beta) \approx 0, \text{ for } |n| \geq 2$$

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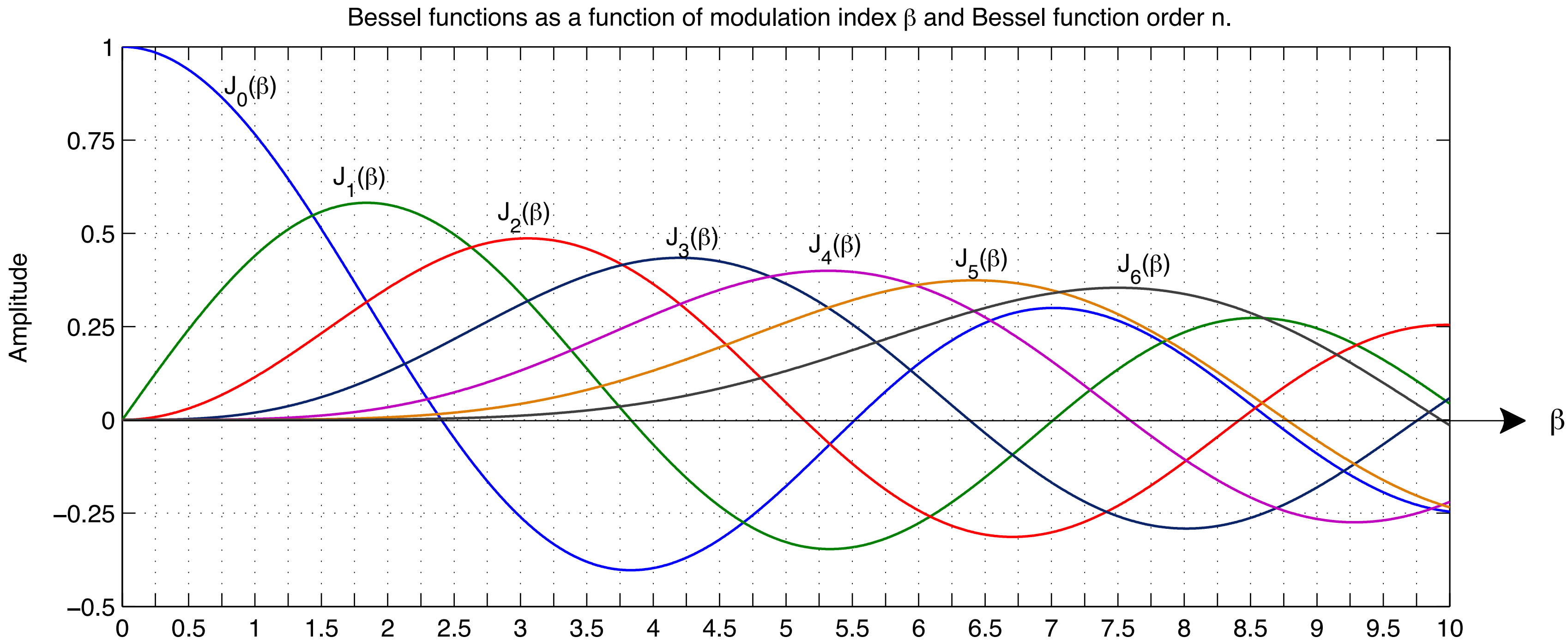
- **Normalization:** $$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

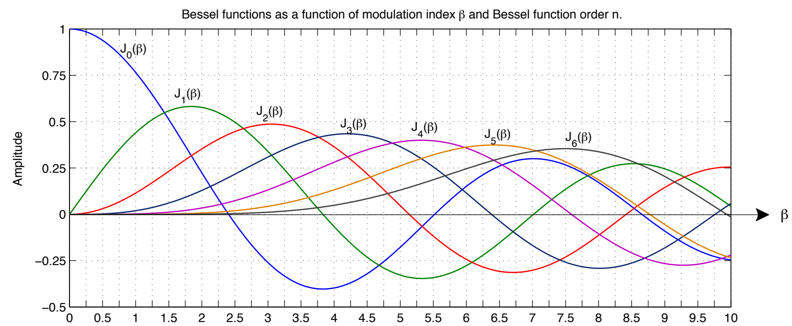
Properties of Bessel functions

- **Well-tabulated.**



Properties of Bessel functions

- **Bessel functions are generated by differential equations.**



Bessel functions are frequently encountered in a number of engineering problems. They are the solutions to the differential equation (e.g. wave equation, heat transfer equation):

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - \nu^2) = 0$$

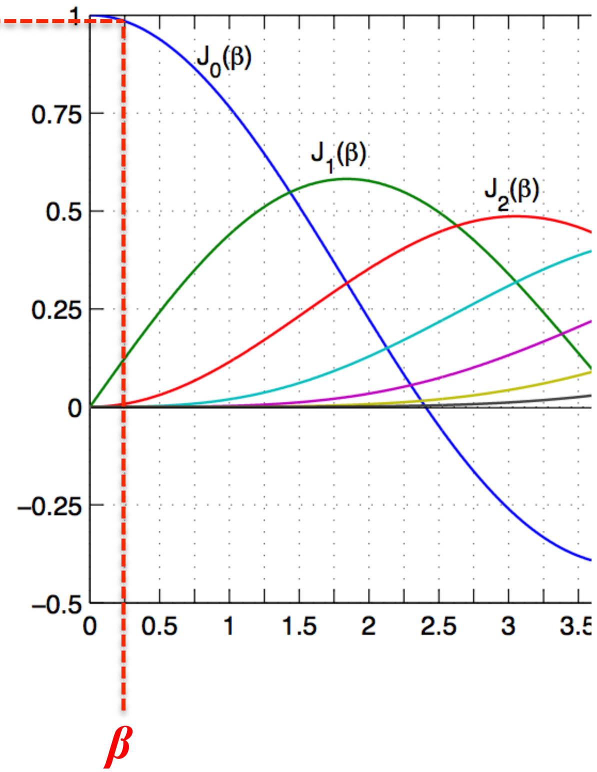
Remember that "sin" and "cos" are also solutions to the differential equation:

$$\frac{d^2 y}{dx^2} + \mu^2 y = 0$$

FM Bandwidth estimation: Single tone modulation

NBFM $(\beta < 0.3)$

$J_0(\beta) \approx 1$



FM Bandwidth estimation: Single tone modulation

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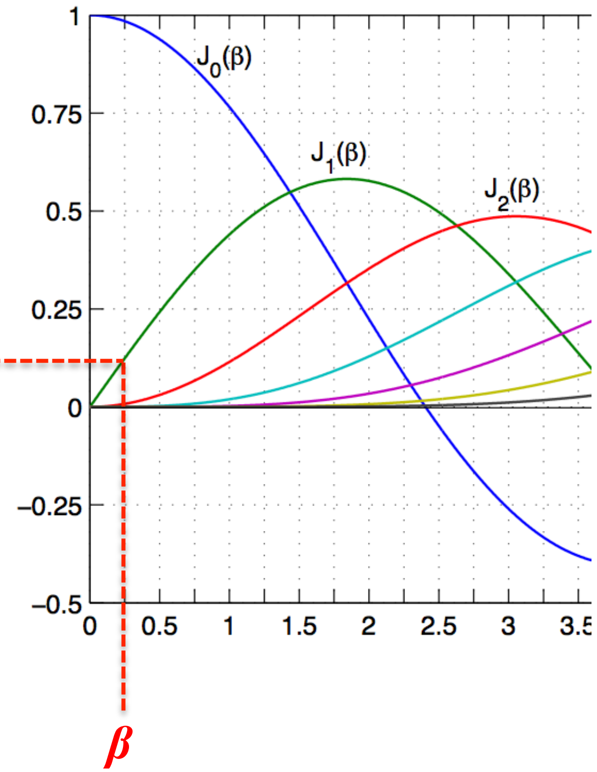

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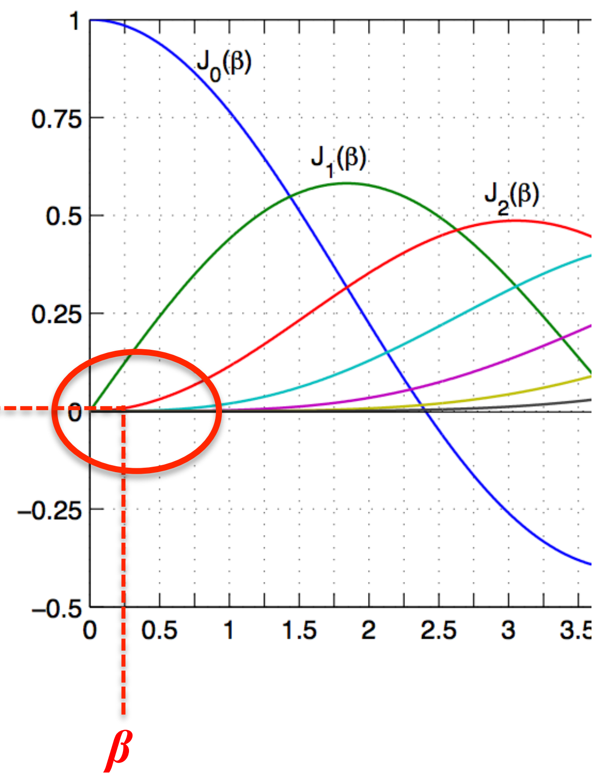

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$$\varphi_{\mathrm{FM}}(t) = A_c \sum_n J_n(\beta) \cos(\omega_c + n\omega_m)t$$

becomes:

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carrier | sideband at f_c+f_m | sideband at f_c-f_m

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carrier — sideband at f_c+f_m — sideband at f_c-f_m

Transmission bandwidth: $B_T \approx 2 f_m$

FM Bandwidth estimation: Single tone modulation

WBFM

- Infinite number of sidebands at $f_c \pm n f_m$

FM Bandwidth estimation: Single tone modulation

WBFM

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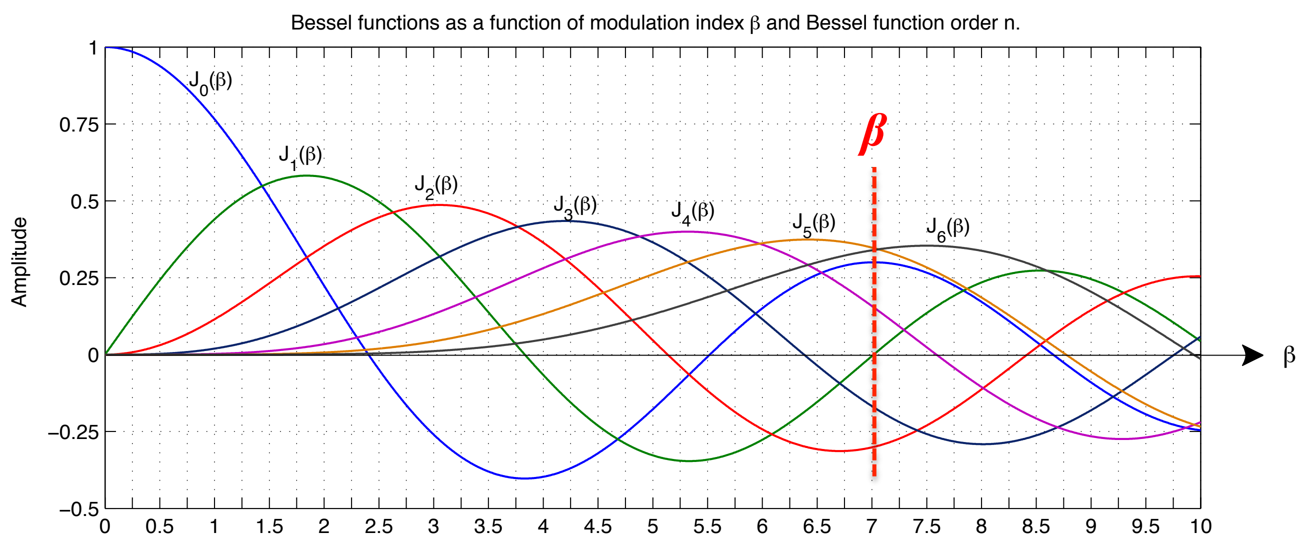
- Magnitudes of sidebands decreases with increasing n, i.e., $|f| \gg f_c$



FM Bandwidth estimation: Single tone modulation

WBFM

- Infinite number of sidebands at $f_c \pm nf_m$
- Magnitudes of sidebands decreases with **increasing** n, i.e., $|f| \gg f_c$
- The power in $\varphi_{\mathrm{FM}}(t)$ is contained within **a finite bandwidth**.

FM Bandwidth estimation: Single tone modulation

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- **Total signal power**

$$\overline{\varphi^2_{\text{FM}}(t)} = \frac{A_c^2}{2} \sum_n J_n^2(\beta) = \frac{A_c^2}{2}$$

FM Bandwidth estimation: Single tone modulation

WBFM

- **Total signal power**

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FM Bandwidth estimation: Single tone modulation

WBFM

- **Total signal power**

$$\overline{\varphi_{\mathrm{FM}}^2(t)} = \frac{A_c^2}{2} \sum_n J_n^2(\beta) = \frac{A_c^2}{2}$$

We could have easily obtained this result from:

$$\varphi_{\mathrm{FM}}(t) = A_c \cos\left(\omega_c t + K_f \int_0^t m(\lambda) d\lambda\right)$$

FM Bandwidth estimation: Single tone modulation

WBFM

- **Total signal power**

$$\overline{\varphi_{\mathrm{FM}}^2(t)} = \frac{A_c^2}{2}\sum_n J_n^2(\beta) = \frac{A_c^2}{2}$$

$$= \textbf{constant}$$

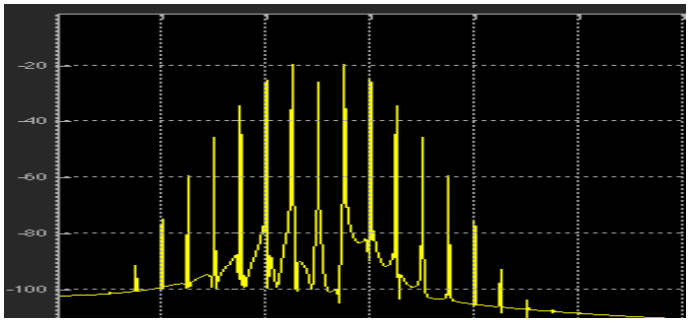
$$= P_{carrier} + P_{sideband}$$

Distribution of the constant total signal power among the carrier and the sidebands will change as a function of the modulation index β.

FM Bandwidth estimation: Single tone modulation

WBFM

Changing the modulation index $\beta = \Delta f / f_m$



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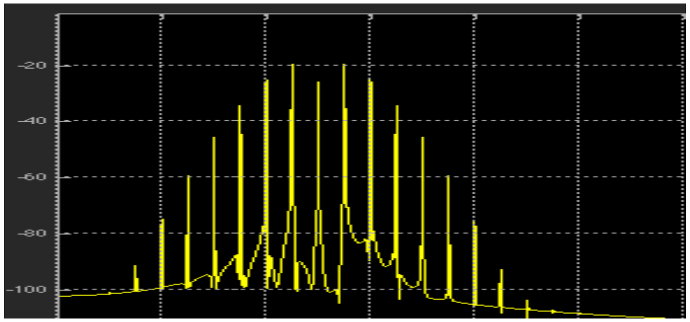


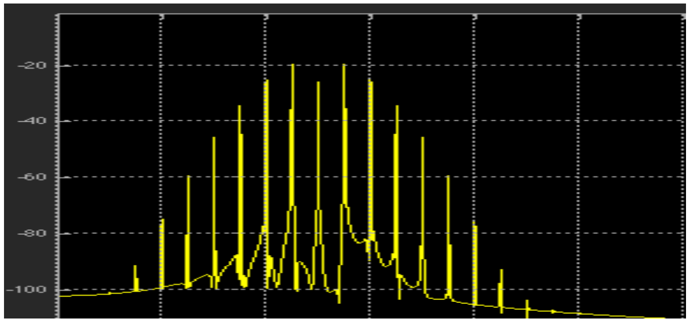
$\Delta f =$ constant   change f_m

- $\beta \uparrow$ with $f_m \downarrow$
- more spectral lines added.

FM Bandwidth estimation: Single tone modulation

WBFM

Changing the modulation index $\beta = \Delta f / f_m$



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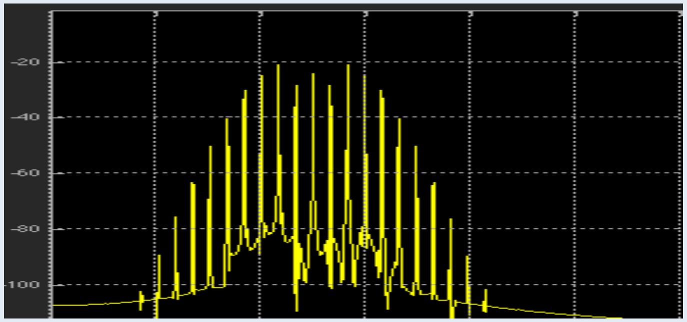

FM Bandwidth estimation: Single tone modulation

WBFM

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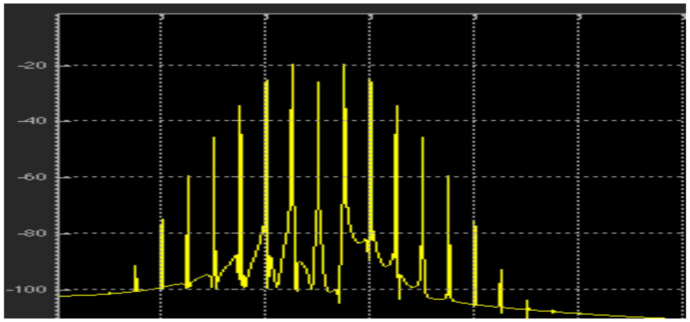
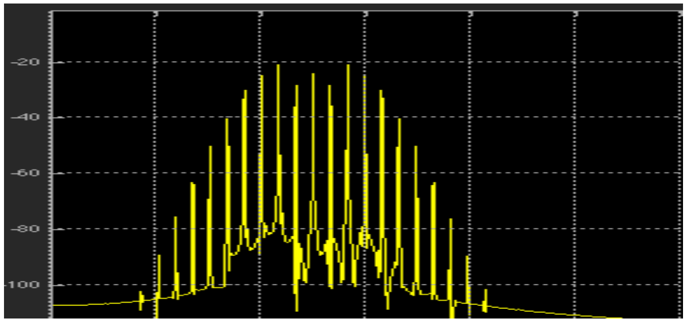


Δf = constant — change f_m

- $\beta \uparrow$ with $f_m \downarrow$
- more spectral lines added.



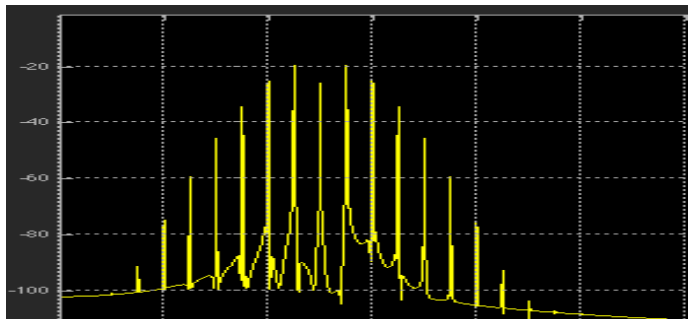
change Δf — f_m = constant

- $\beta \uparrow$ with $\Delta f \uparrow$
- more spectral lines at constant spacing.

FM Bandwidth estimation: Single tone modulation

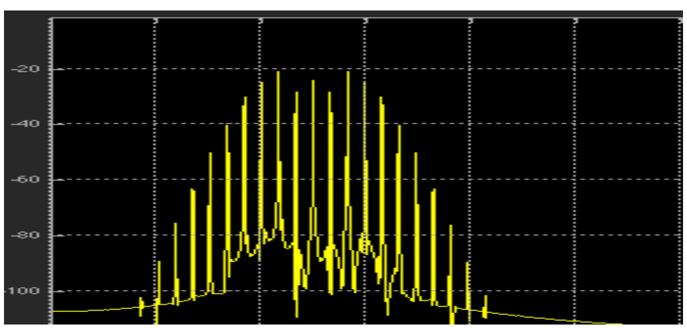
WBFM

Changing the modulation index $\beta = \Delta f / f_m$



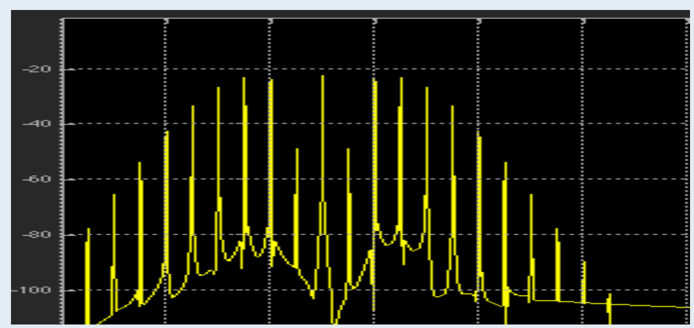
Δf = constant — change f_m

- $\beta \uparrow$ with $f_m \downarrow$
- more spectral lines added.



change Δf — f_m = constant

- $\beta \uparrow$ with $\Delta f \uparrow$
- more spectral lines at constant spacing.



FM Bandwidth estimation: Single tone modulation

WBFM

How many sideband are significant to determine B_T?

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A common rule:

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- Unmodulated carrier .... : $A_c J_0(0)$
- Sideband terms .... : $A_c J_n(\beta)$

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- Unmodulated carrier .... : $A_c J_0(0)$
- Sideband terms .... : $A_c J_n(\beta)$

For the **1-% case** determine the **sideband index n** such that

$$A_c J_n(\beta) \,|\, > \,|\, A_c J_0(0) \,|\, 10^{-2}$$

$$|\, J_n(\beta) \,|\, > \,|\, J_0(0) \,|\, 10^{-2}$$

$$|\, J_n(\beta) \,|\, > 10^{-2} \qquad \text{since} \qquad J_0(0) = 1$$

FM Bandwidth estimation: Single tone modulation

WBFM

How many sideband are significant to determine B_T?

If n_{max} is the largest value of the sideband index satisfying the requirement:

$$|J_n(\beta)| > 10^{-2}$$

then we approximate the transmission bandwidth as

$$B_T = 2\, n_{max} f_m$$

FM Bandwidth estimation: Single tone modulation

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then we approximate the transmission bandwidth as

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We use table of Bessel functions to determine the number of sidebands.

FM Bandwidth estimation: Single tone modulation

1-% rule

Table of Bessel Functions as a function of modulation index β and Bessel function order n.

β	$J_0(\beta)$	$J_1(\beta)$	$J_2(\beta)$	$J_3(\beta)$	$J_4(\beta)$	$J_5(\beta)$	$J_6(\beta)$	$J_7(\beta)$	$J_8(\beta)$	$J_9(\beta)$
0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.2	0.99	0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.4	0.96	0.20	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.91	0.29	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.8	0.85	0.37	0.08	0.01	0.00	0.00	0.00	0.00	0.00	0.00
1.0	0.77	0.44	0.11	0.02	0.00	0.00	0.00	0.00	0.00	0.00
1.2	0.67	0.50	0.16	0.03	0.01	0.00	0.00	0.00	0.00	0.00
1.4	0.57	0.54	0.21	0.05	0.01	0.00	0.00	0.00	0.00	0.00
1.6	0.46	0.57	0.26	0.07	0.01	0.00	0.00	0.00	0.00	0.00
1.8	0.34	0.58	0.31	0.10	0.02	0.00	0.00	0.00	0.00	0.00
2.0	0.22	0.58	0.35	0.13	0.03	0.01	0.00	0.00	0.00	0.00
2.2	0.11	0.56	0.40	0.16	0.05	0.01	0.00	0.00	0.00	0.00
2.4	0.00	0.52	0.43	0.20	0.06	0.02	0.00	0.00	0.00	0.00
2.6	-0.10	0.47	0.46	0.24	0.08	0.02	0.01	0.00	0.00	0.00
2.8	-0.19	0.41	0.48	0.27	0.11	0.03	0.01	0.00	0.00	0.00
3.0	-0.26	0.34	0.49	0.31	0.13	0.04	0.01	0.00	0.00	0.00
3.2	-0.32	0.26	0.48	0.34	0.16	0.06	0.02	0.00	0.00	0.00
3.4	-0.36	0.18	0.47	0.37	0.19	0.07	0.02	0.01	0.00	0.00
3.6	-0.39	0.10	0.44	0.40	0.22	0.09	0.03	0.01	0.00	0.00
3.8	-0.40	0.01	0.41	0.42	0.25	0.11	0.04	0.01	0.00	0.00
4.0	-0.40	-0.07	0.36	0.43	0.28	0.13	0.05	0.02	0.00	0.00
4.2	-0.38	-0.14	0.31	0.43	0.31	0.16	0.06	0.02	0.01	0.00
4.4	-0.34	-0.20	0.25	0.43	0.34	0.18	0.08	0.03	0.01	0.00
4.6	-0.30	-0.26	0.18	0.42	0.36	0.21	0.09	0.03	0.01	0.00
4.8	-0.24	-0.30	0.12	0.40	0.38	0.23	0.11	0.04	0.01	0.00
5.0	-0.18	-0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02	0.01
5.2	-0.11	-0.34	-0.02	0.33	0.40	0.29	0.15	0.07	0.02	0.01
5.4	-0.04	-0.35	-0.09	0.28	0.40	0.31	0.18	0.08	0.03	0.01
5.6	0.00	-0.33	-0.15	0.23	0.39	0.33	0.20	0.09	0.04	0.01
5.8	0.09	-0.31	-0.20	0.17	0.38	0.35	0.22	0.11	0.05	0.02
6.0	0.15	-0.28	-0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02

FM Bandwidth estimation: Single tone modulation

10-% rule

Table of Bessel Functions as a function of modulation index β and Bessel function order n.

β	$J_0(\beta)$	$J_1(\beta)$	$J_2(\beta)$	$J_3(\beta)$	$J_4(\beta)$	$J_5(\beta)$	$J_6(\beta)$	$J_7(\beta)$	$J_8(\beta)$	$J_9(\beta)$
0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.2	0.99	0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.4	0.96	0.20	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.91	0.29	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.8	0.85	0.37	0.08	0.01	0.00	0.00	0.00	0.00	0.00	0.00
1.0	0.77	0.44	0.11	0.02	0.00	0.00	0.00	0.00	0.00	0.00
1.2	0.67	0.50	0.16	0.03	0.01	0.00	0.00	0.00	0.00	0.00
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3.0	-0.26	0.34	0.49	0.31	0.13	0.04	0.01	0.00	0.00	0.00
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4.2	-0.38	-0.14	0.31	0.43	0.31	0.16	0.06	0.02	0.01	0.00
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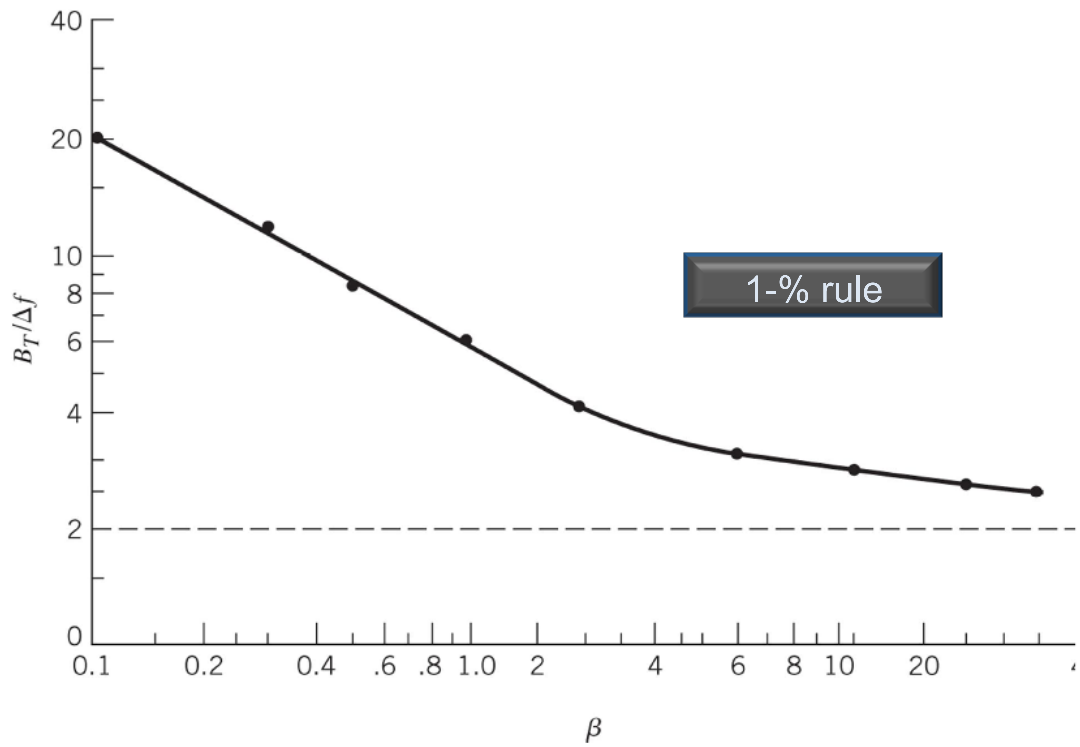
FM Bandwidth estimation: Single tone modulation

The transmission bandwidth B_T calculated using the x-% rule can also be presented in the form of a **universal curve** by normalizing the results obtained from the Bessel function tables shown in the previous slides with respect to the frequency deviation Δf and then plotting it versus β.



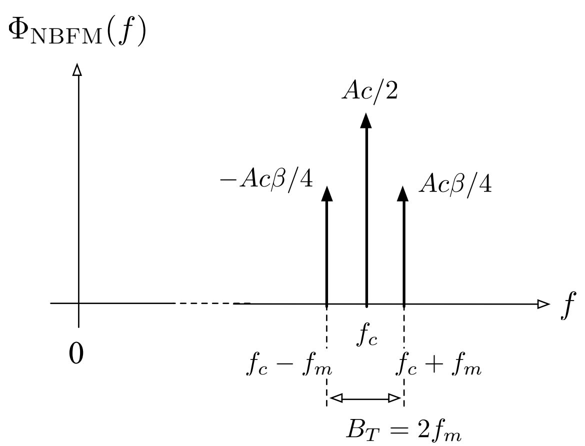
NBFM Bandwidth estimation: Summary

Single Tone Modulation $m(t) = A_m \cos 2\pi f_m t$

$$\beta = \frac{\Delta f}{f_m} = \frac{K_f A_m / 2\pi}{f_m}$$

with $\beta < 0.3$

$$\varphi_{\text{NBFM}}(t) \approx A_c \cos \omega_c t - A_c \beta \sin \omega_m t \sin \omega_c t$$

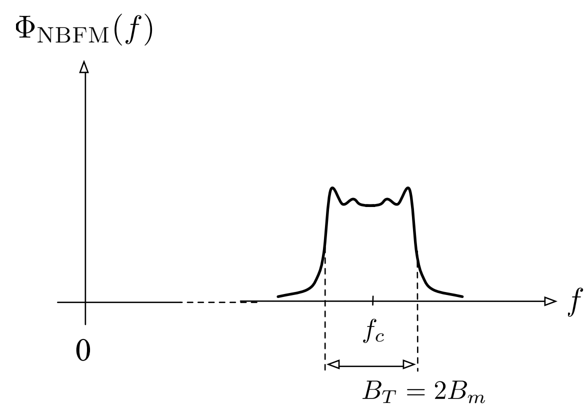


Arbitrary $m(t)$ $m(t)$ bandlimited B_m

$$\mathcal{D} = \frac{\Delta f}{B_m} = \frac{K_f m_p / 2\pi}{B_m}$$

with $\mathcal{D} < 0.3$ and $m_p = \max |m(t)|$

$$\varphi_{\text{NBFM}}(t) \approx A_c \cos \omega_c t - A_c \left[K_f \int m(\lambda) d\lambda \right] \sin \omega_c t$$



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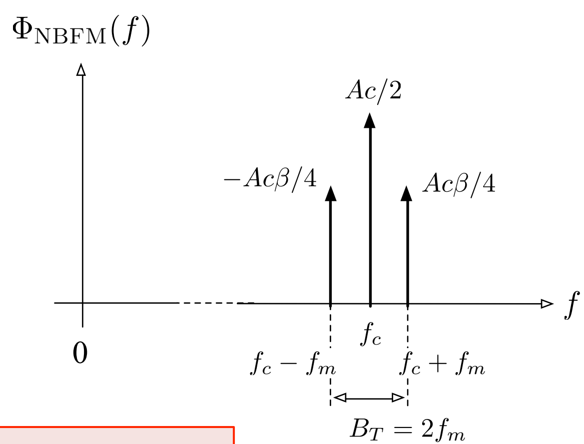

$$B_T \approx 2f_m$$

Arbitrary $m(t)$	$m(t)$ bandlimited B_m

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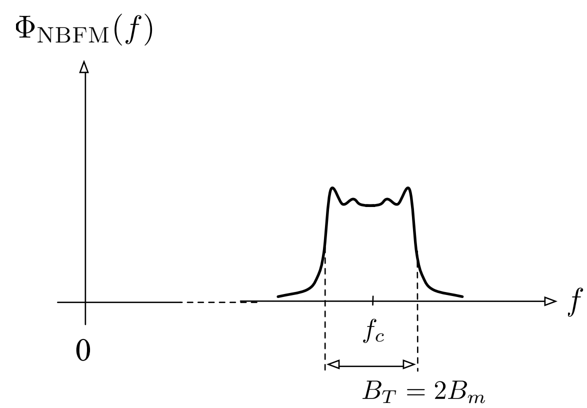


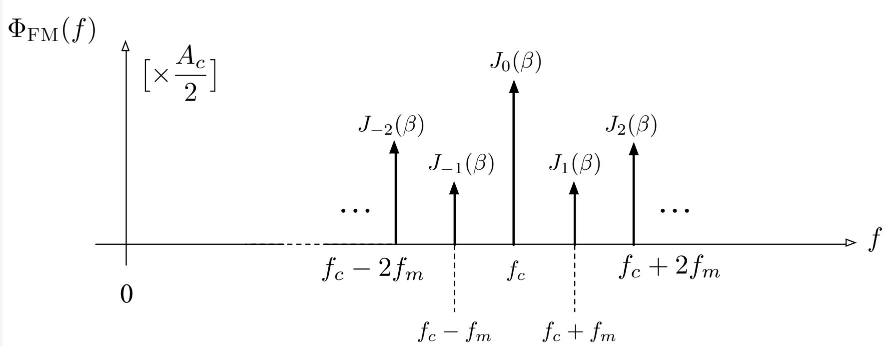
$$B_T \approx 2B_m$$

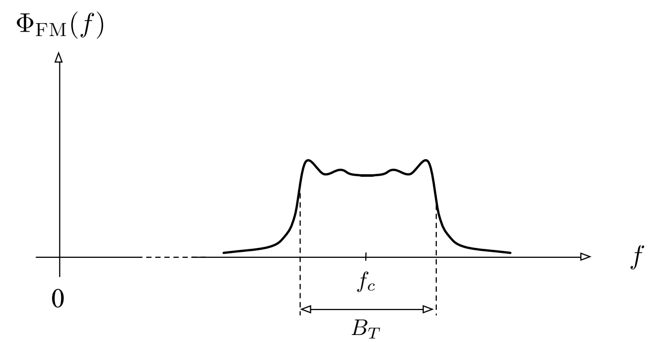
WBFM Bandwidth estimation: Summary

Single Tone Modulation	$m(t) = A_m \cos 2\pi f_m t$	Arbitrary $m(t)$	$m(t)$ bandlimited B_m

$$\varphi_{\mathrm{FM}}(t) = A_c \sum_n J_n(\beta) \cos(\omega_c + n\omega_m)t$$

$$\varphi_{\mathrm{FM}}(t) = A_c \cos\left(\omega_c t + K_f \int_0^t m(\lambda)d\lambda\right)$$





Estimate B_T using:

- 1-% or 10-% rule as applicable (or the corresponding the universal curve), or
- Carson's Rule.

Estimate B_T using:

- Carson's rule, or
- the universal curve.

FM Bandwidth estimation: Further comments

Comments on the Universal Curve:

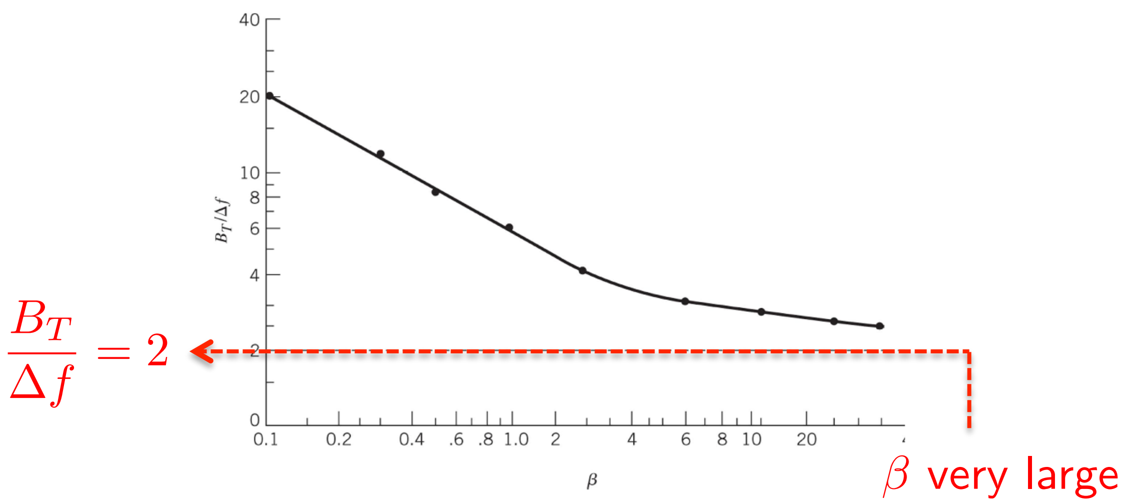
- For arbitrary signals we can continue using the universal curve by replacing:
  - Modulation index β with the **deviation index** $\mathcal{D}$.
  - Modulating frequency f_m with **signal bandwidth** B_m.

FM Bandwidth estimation: Further comments

Comments on the Universal Curve:

- For arbitrary signals we can continue using the universal curve by replacing:
  - Modulation index β with the **deviation index** $\mathcal{D}$.
  - Modulating frequency f_m with **signal bandwidth** B_m.

- From the universal curve we observe that $B_T/\Delta f \approx 2$ as $\beta \uparrow$ (or $\mathcal{D} \uparrow$)

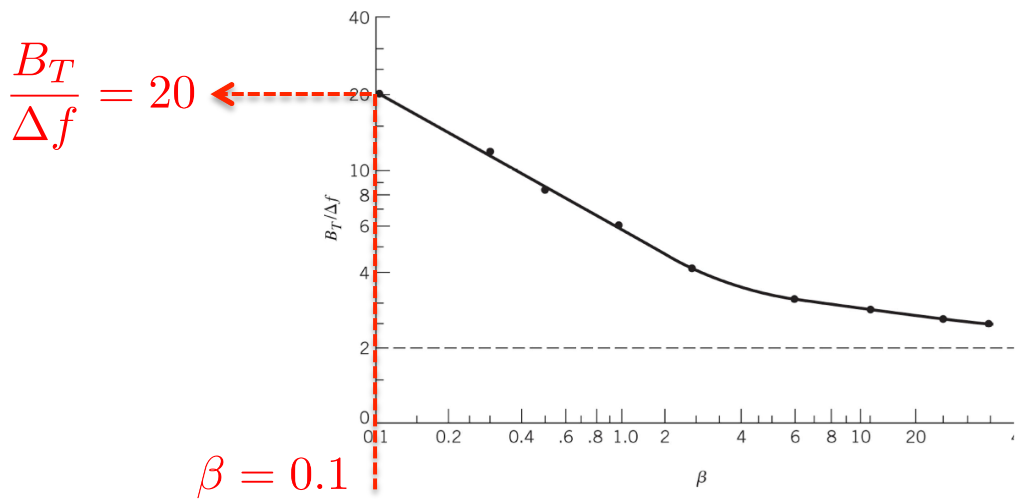


Therefore, for large β (or $\mathcal{D}$) we have: $B_T \approx 2\Delta f$

FM Bandwidth estimation: Further comments

Comments on the Universal Curve:

- From the universal curve we observe that $B_T/\Delta f \approx 20$ for $\beta = 0.1$



Therefore, for large $\beta = 0.1$ (or $\mathcal{D} = 0.1$) we have:

$$\left.\frac{B_T}{\Delta f}\right|_{\beta=0.1} = 20 \quad \Rightarrow \quad B_T = 20 \times \Delta f\Big|_{\beta=0.1} = 20 \times \beta f_m\Big|_{\beta=0.1} = 20 \times (0.1) f_m = 2 f_m$$

$$B_T \approx 2 f_m$$

FM Bandwidth estimation: Further comments

Comments on Carson's Rule:

Transmission bandwidth

NBFM

$B_T \approx 2B_m$

increasing β

WBFM

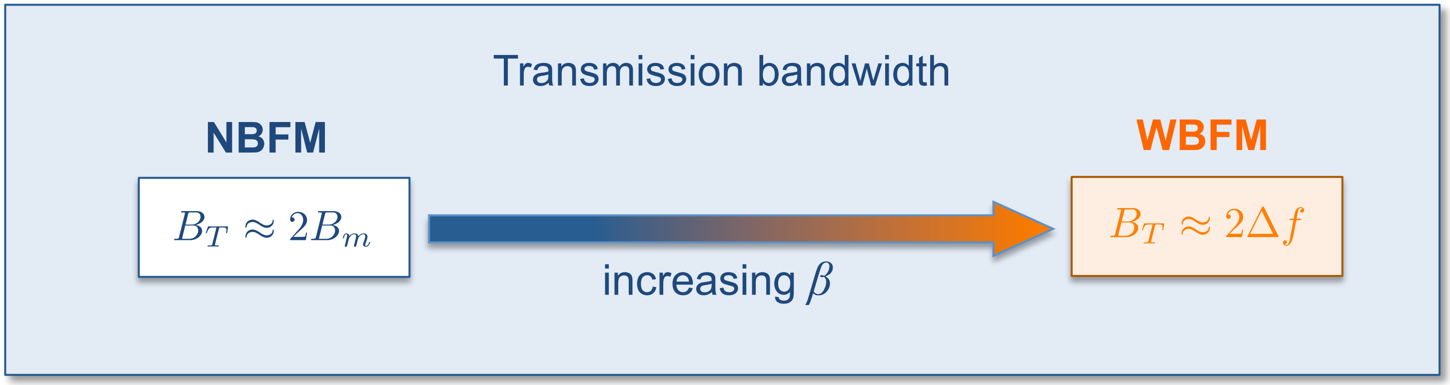
$B_T \approx 2\Delta f$

For intermediate values of β (or $\mathcal{D}$): use **Carson's Rule**

$$B_{\mathrm{FM}} \approx 2B_m(1+\beta) = 2(B_m + \Delta f)$$

FM Bandwidth estimation: Further comments

Comments on Carson's Rule:



- B_T estimated using Carson's Rule approaches correct limits as $\beta \to 0$ and $\beta \uparrow$.
- B_T(Carson) < B_T(1%-rule) with maximum error for $\beta \approx 1$.
- Carson's Rule displays the effects of the two mechanism in the generation of FM signals.

FM Bandwidth estimation: An example

North America commercial FM broadcasting:

- Maximum allowed frequency deviation $\Delta f = 75$ kHz.
- Bandwidth of modulating signal $B_m = 15$ kHz.

Bandwidth estimate from Carson's Rule

$$B_T \approx 2(B_m + \Delta f) = 2(15 + 75) = 180 \text{ kHz}$$

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Bandwidth estimate from Carson's Rule

$$B_T \approx 2(B_m + \Delta f) = 2(15 + 75) = 180 \text{ kHz}$$

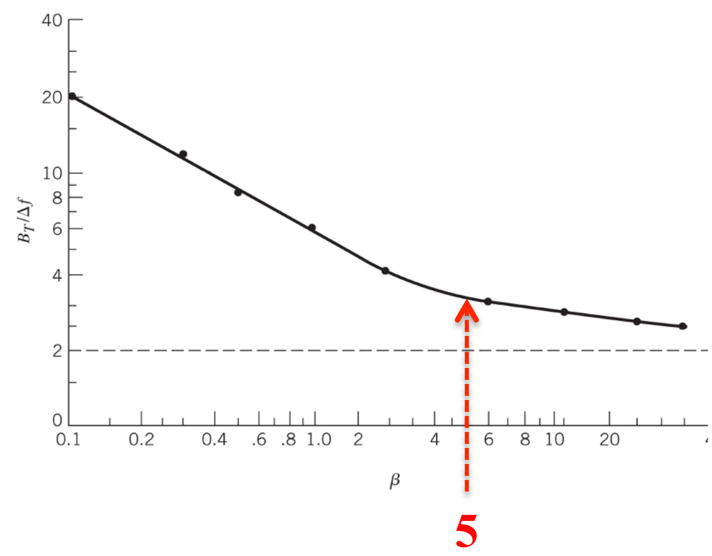
Bandwidth estimate from 1-% Rule using the Universal Curve

Calculate the deviation ratio:

$$\mathcal{D} = \frac{\Delta f}{B_m} = \frac{75}{15} = 5$$

FM Bandwidth estimation: An example

Bandwidth estimate from 1-% Rule using the Universal Curve

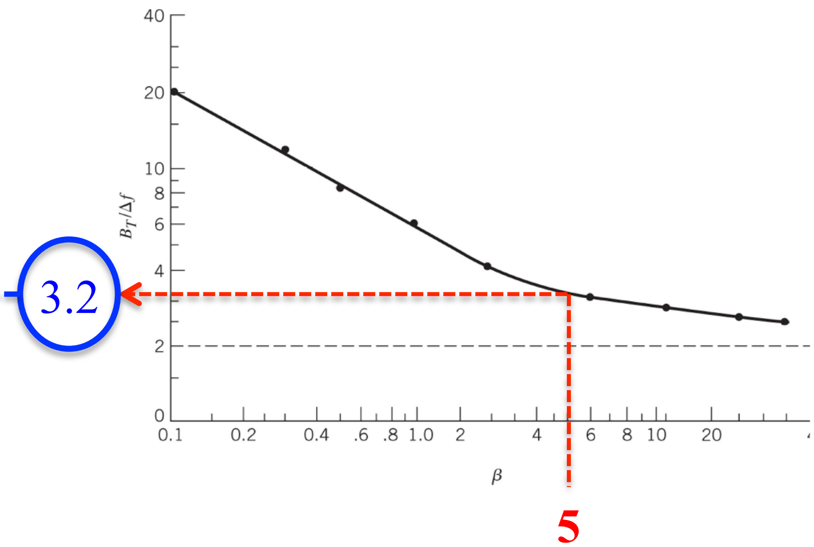


$$\mathcal{D} = \frac{\Delta f}{B_m} = \frac{75}{15} = 5$$

FM Bandwidth estimation: An example

Bandwidth estimate from 1-% Rule using the Universal Curve

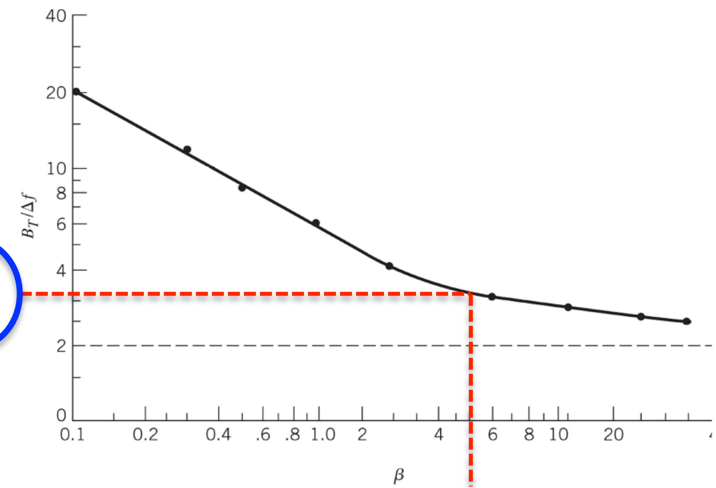
$$\frac{B_T}{\Delta f} \approx \mathbf{3.2}$$



$$\mathcal{D} = \frac{\Delta f}{B_m} = \frac{75}{15} = 5$$

FM Bandwidth estimation: An example

Bandwidth estimate from 1-% Rule using the Universal Curve



$$\frac{B_T}{\Delta f} \approx \mathbf{3.2}$$

3.2

$$B_T \approx \mathbf{3.2}\,\Delta f = \mathbf{3.2} \times 75 = 240 \text{ kHz}$$

5

$$\mathcal{D} = \frac{\Delta f}{B_m} = \frac{75}{15} = \mathbf{5}$$

FM Bandwidth estimation: An example

Bandwidth estimate from 1-% Rule using the Universal Curve

Observe that the result from the universal curve could have also been obtained from the table of Bessel functions by replacing β with $\mathcal{D}$ and f_m with B_m.

Table of Bessel Functions as a function of modulation index β and Bessel function order n.

β	$J_0(\beta)$	$J_1(\beta)$	$J_2(\beta)$	$J_3(\beta)$	$J_4(\beta)$	$J_5(\beta)$	$J_6(\beta)$	$J_7(\beta)$	$J_8(\beta)$	$J_9(\beta)$
0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.2	0.99	0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.4	0.96	0.20	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.91	0.29	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.8	0.85	0.37	0.08	0.01	0.00	0.00	0.00	0.00	0.00	0.00
1.0	0.77	0.44	0.11	0.02	0.00	0.00	0.00	0.00	0.00	0.00
1.2	0.67	0.50	0.16	0.03	0.01	0.00	0.00	0.00	0.00	0.00
1.4	0.57	0.54	0.21	0.05	0.01	0.00	0.00	0.00	0.00	0.00
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3.0	-0.26	0.34	0.49	0.31	0.13	0.04	0.01	0.00	0.00	0.00
3.2	-0.32	0.26	0.48	0.34	0.16	0.06	0.02	0.00	0.00	0.00
3.4	-0.36	0.18	0.47	0.37	0.19	0.07	0.02	0.01	0.00	0.00
3.6	-0.39	0.10	0.44	0.40	0.22	0.09	0.03	0.01	0.00	0.00
3.8	-0.40	0.01	0.41	0.42	0.25	0.11	0.04	0.01	0.00	0.00
4.0	-0.40	-0.07	0.36	0.43	0.28	0.13	0.05	0.02	0.00	0.00
4.2	-0.38	-0.14	0.31	0.43	0.31	0.16	0.06	0.02	0.01	0.00
4.4	-0.34	-0.20	0.25	0.43	0.34	0.18	0.08	0.03	0.01	0.00
4.6	-0.30	-0.26	0.18	0.42	0.36	0.21	0.09	0.03	0.01	0.00
4.8	-0.24	-0.30	0.12	0.40	0.38	0.23	0.11	0.04	0.01	0.00
5.0	-0.18	-0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02	0.01
5.2	-0.11	-0.34	-0.02	0.33	0.40	0.29	0.15	0.07	0.02	0.01
5.4	-0.04	-0.35	-0.09	0.28	0.40	0.31	0.18	0.08	0.03	0.01
5.6	0.00	-0.33	-0.15	0.23	0.39	0.33	0.20	0.09	0.04	0.01
5.8	0.09	-0.31	-0.20	0.17	0.38	0.35	0.22	0.11	0.05	0.02
6.0	0.15	-0.28	-0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02

$\mathcal{D} = 5$

$J_8(\beta)$

$n_{max} = 8$

$$B_T = 2\, n_{max} B_m = 2\,(8) \times (15) = 240 \text{ kHz}$$

FM Bandwidth estimation: An example

Carson's Rule	1-% Rule using the Universal Curve
$B_T = 180$ kHz	$B_T = 240$ kHz

- In practice for a **bandwidth of 200 kHz** is allocated to each FM transmitter.
- **Carson's Rule underestimates** B_T by 10%.
- 1%-rule using the **universal curve overestimates** B_T by 20%.